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Exact Solutions for the Transient Densities of Continuous-Time Markov Switching Models - With an Application to the Poisson Multifractal Model*

Thomas Lux†

Abstract

This paper shows how exact solutions for the transient density of a large class of continuous-time Markov switching models can be obtained. We illustrate the pertinent approach for both simple diffusion models with a small number of regimes as well as for the more complicated so-called Poisson multifractal model introduced by Calvet and Fisher (2001) with an arbitrarily large number of regimes. Our results can be immediately applied as well to various popular Markov-switching models in financial economics. Closed-form solutions provide for the possibility of exact maximum likelihood estimation for discretely sampled Markov-switching diffusions and also facilitate the use of such models in applied tasks such as option pricing and portfolio management.

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1 Introduction

Since Hamilton's (1989) introduction of a regime-switching process for the statistical analysis of the different phases of the business cycle, Markov-switching models have become a very widespread and popular tool in empirical economics and empirical finance. While Hamilton's approach had been cast in discrete time, continuous-time models with Markov switching regimes have been developed subsequently particularly for applications in finance. Examples include interest rate models (Smith, 2002; Dai and Singleton, 2003; Choi, 2009), and the analysis of option pricing and hedging under regime-switching dynamics of the underlying (Barone-Adesi and Whaley, 1987; Guo and Zhang, 2004; Elliott et al., 2007, 2011; Fuh et al., 2012).

When estimating such models, the typical problem encountered is the same as with continuous-time models more generally: that one observes only discrete realizations of the underlying instantaneous dynamics, and that safe for very simple cases, such objects as the conditional density of an observation at some $t + \Delta t$ given the value at time t are unknown (Aït-Sahalia, 2007). Hence, the absence of closed-form solutions for conditional densities often makes exact likelihood estimation unfeasible. In the case of Markov-switching diffusions, the added layer of complexity due to regime changes has so far prevented likelihood inference even in those cases where the stochastic differential equations activated in different regimes all possess closed-form solutions of their densities. As a result, a large literature has developed that pursues alternative avenues for parameter estimation of Markov-switching diffusions. For instance, quasi-ML estimation is used by Smith (2002) for a discretized version of a Markov-switching stochastic volatility model. Choi (2009) uses Hermite polynomial expansions for the individual regimes of a Markov-switching model of interest rate dynamics together with a discretized Hamilton (1994) algorithm for updating conditional regime probabilities, Metzner et al. (2007)[1] use an EM algorithm, Elliott et al. (2008) develop moment-based estimators, and a relatively large strand of literature has advanced Markov chain Monte Carlo methods for such processes, cf. Liechty and Roberts (2001), Hahn et al. (2010), among others.

In this paper, we show that for a large class of regime-switching models, closed-form con-

[1] This paper also provides some information on the use of Markov switching diffusions in other research areas, particularly in molecular dynamics.

ditional densities might be obtained in a relatively straightforward way using an appropriate *ansatz*. Basically, in all those cases where the components (individual regimes) follow a diffusion process with a known density (not necessarily the same for all regimes), the density of the entire process can also be determined explicitly. This broad applicability will become already fully evident when dealing with the simple case of a pure diffusion process with two regimes that we consider in sec. 2. In sec. 3, we apply the same approach to a much more complicated model with a large number of volatility states: The Poisson multifractal asset-pricing model introduced by Calvet and Fisher (2001). Discretized versions of this new model have been found to improve upon standard volatility models in a number of applications (Calvet and Fisher, 2004, 2006; Lux, 2008; Lux and Kaizoji, 2007) and have also been adapted for modelling inter-trade durations (Chen et al., 2013). Recent research also shows very successful applications of continuous-time multifractal models for term structure modelling (Calvet et al., 2012) and option pricing (Calvet et al., 2013). Essentially, the 'multifractal' approach appears to provide a very versatile family of models to capture a hierarchical multi-component structure of a dynamic process in a parsimonious and robust way. We show that the continuous-time multifractal model can be interpreted as a regime switching diffusion with a large number of states, and that, with some effort, the approach introduced in sec. 2 allows to establish its transient density in closed form. Given the promising results of the previous work on this new class of time-series models, the possibility of working with closed-form densities should be particularly useful. Sec. 4 will present an illustration of ML estimation based on the closed-form density of the Poisson multifractal model, and sec. 5 concludes.

2 Exact, closed form solutions for simple regime-switching diffusion models

We are interested in continuous-time regime-switching diffusion processes with a latent Markov chain s_t that determines the choice of the pertinent active regimes with their regime-dependent drift and diffusion functions:

$$dx_t = \mu\left(x_t, s_t; \theta\right) dt + \sigma\left(x_t, s_t; \theta\right) dW_t \tag{1}$$

with x_t a univariate variable following a regime-switching diffusion process, $\mu\left(\cdot\right)$ the drift and $\sigma\left(\cdot\right)$ the diffusion functions, and θ a vector of parameters. s_t follows a continuous-time first-order Markov chain with n states with matrix Q of intensity parameters. The entries q_{ij} of Q are the intensities, i.e. rates of probability, with which the process switches from state i to state j, and $q_{ii} = -\sum_{i \neq j} q_{ij}$.

For each regime, all standard conditions will be assumed to be satisfied to allow characterization of the time development of the pertinent transient pdf by the usual infinitesimal generator and forward Kolmogorov (Fokker-Planck) equation:

$$\frac{\partial f_k(x, \theta; t)}{\partial f_k} = -\frac{\partial}{\partial x}[\mu_k(x; \theta)\ f_k(x, \theta; t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma_k^2(x; \theta)\ f_k(x, \theta; t)] \tag{2}$$

with $\mu_k(\cdot)$, $\sigma_k(\cdot)$ the drift and diffusion functions activated for $s_t = k$ (i.e. all the processes active in different regimes are well-behaved).

For the compound process, stochastic calculus allows to determine the dynamic solution of the joint densities $f(x, s, \theta; t)$ by a system of Fokker-Planck equations (cf. Mao and Yuan, 2006; Yin and Zhu, 2010):

$$\frac{\partial f(x, s; \theta, t)}{\partial t} = -\frac{\partial}{\partial x}[\mu(x, s; \theta)\ f(x, s; \theta, t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma(x, s; \theta)\ f(x, s; \theta, t)] + Q \cdot f(x, s; \theta, t) \tag{3}$$

There has been a sourge of recent interests in such regime-switching (or hybrid) diffusion processes. Basic results on the existence of an ergodic distribution and the convergence of the transient dynamics to the stationary distribution are provided in Skorokhod (1989, II, §2) Surveys of recent mathematical results can be found in Mao and Yuan (2006) and Yin and Zhu (2010). The system of Fokker-Planck equations characterizing the transient densities has been used to derive stationary distributions by Mao et al. (2005). Kauschke (2010) applies their results to an agent-based behavioural finance model. Despite a variety of applications in finance, molecular dynamics, and communication technology, to the best of our knowledge,

available literature has never attempted to derive closed-form solutions for the time evolution of the joint density of such processes. One contribution of our paper will be to provide an avenue towards such exact solutions for a large class of regime-switching models.

To illustrate how to obtain closed-form solutions to the transient density of a regime-switching diffusion, we first consider the simple case of a pure diffusion process with two regimes:

$$dx_t = \sigma(x_t, s_t)dW_t \tag{4}$$

with $\sigma(.,.) = \sigma_1$ if $s_t = 1$ and $\sigma(.,.) = \sigma_2$ if $s_t = 2$. The continuous-time Markov chain s_t is governed by the infinitesimal matrix of intensity parameters q_{ij}:

$$Q = \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix} = \begin{pmatrix} -q_{12} & q_{21} \\ q_{12} & -q_{21} \end{pmatrix}. \tag{5}$$

The intensity parameter q_{ij} gives the rate at which the process switches from state i to state j in the continuous-time limit and it holds that $q_{ij} > 0$ for $i \neq j$. The transient density is, then, defined by the solution to the following system of two linear second-order partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= a_1 \frac{\partial^2 u}{\partial x^2} - q_{12}u + q_{21}w, \\ \frac{\partial w}{\partial t} &= a_2 \frac{\partial^2 w}{\partial x^2} - q_{21}w + q_{12}u. \end{aligned} \tag{6}$$

In (6), we have defined the state-wise densities as $u(x,t) = f(x, s = 1; t)$ and $w(x,t) = f(x, s = 2; t)$ and $a_1 = \frac{1}{2}\sigma_1^2, a_2 = \frac{1}{2}\sigma_2^2$. Following an approach for the solution of similar systems of partial differential equations suggested in different applications (Polyanin and Manzhirov, 2007, sec. T10), we assume that the solution to this problem can be decomposed as follows:

$$u(x,t) = \varphi_1(t)\theta_1(x,t), \quad w(x,t) = \varphi_2(t)\theta_2(x,t). \tag{7}$$

This leads upon substitution to:

$$\begin{aligned}\varphi_1'\theta_1(x,t)+\varphi_1\frac{\partial\theta_1}{\partial t}&=a_1\varphi_1\frac{\partial^2\theta_1}{\partial x^2}-q_{12}\varphi_1(t)\theta_1(x,t)+q_{21}\varphi_2(t)\theta_2(x,t),\\ \varphi_2'\theta_2(x,t)+\varphi_2\frac{\partial\theta_2}{\partial t}&=a_2\varphi_2\frac{\partial\theta_2}{\partial x^2}-q_{21}\varphi_2(t)\theta_2(x,t)+q_{12}\varphi_1(t)\theta_1(x,t).\end{aligned} \tag{8}$$

A solution to this system consists in θ_1 and θ_2 solving the independent partial differential equations:

$$\frac{\partial\theta_1}{\partial t}=a_1\frac{\partial\theta_1^2}{\partial x^2},\frac{\partial\theta_2}{\partial t}=a_2\frac{\partial\theta_2}{\partial x^2}, \tag{9}$$

and $\varphi_1(t)$, $\varphi_2(t)$ being the solutions to:

$$\begin{aligned}\varphi_1'\theta_1&=-q_{12}\varphi_1\theta_1+q_{21}\varphi_2\theta_2,\\ \varphi_2'\theta_2&=-q_{21}\varphi_2\theta_2+q_{12}\varphi_1\theta_1.\end{aligned} \tag{10}$$

Note that the dynamic laws of eq. (9) just represent the evolution of the densities of two Brownian motions for states 1 and 2. Solving eqs. (10) requires slightly more effort. Trying typical solutions of the form $\varphi_1(t)=\alpha_1e^{\lambda_1t}$,$\varphi_2(t)=\alpha_2e^{\lambda_2t}$, one finds that $\lambda_{1,2}$ solve the characteristic equation:

$$\lambda^2+(q_{12}+q_{21})\lambda=0. \tag{11}$$

Hence, there are two eigenvalues solving this system: $\lambda_1=0$ and $\lambda_2=-(q_{12}+q_{21})$. The time-dependent solutions of $\varphi_1(t)$ and $\varphi_2(t)$ can be written as:

$$\begin{aligned}\varphi_1(t)&=A_1\frac{q_{21}}{q_{12}}-A_2e^{\lambda_2t},\\ \varphi_2(t)&=A_1+A_2e^{\lambda_2t}.\end{aligned} \tag{12}$$

Where the first term on the right-hand side corresponds to the solution for the root $\lambda_1=0$, and A_1 and A_2 are constants of integration to be fixed by appropriate initial or boundary conditions.

In our case, the constants may be obtained in most applications from the probabilities of observing state 1 or 2 at time $t = 0$. Denoting by p_0 the probability of state 1 at time t=0 we obtain the following equations for determination of A_1 and A_2:

$$\begin{aligned} \varphi_1(0) &= A_1 \frac{q_{21}}{q_{12}} - A_2 = p_0, \\ \varphi_2(0) &= A_1 + A_2 = 1 - p_0. \end{aligned} \tag{13}$$

We obtain: $A_1 = \frac{q_{12}}{q_{12}+q_{21}}$, $A_2 = (1 - p_0)\frac{q_{21}-q_{12}}{q_{21}+q_{12}}$.

Inserting in (12), we find the closed-form solutions:

$$\begin{aligned} \varphi_1(t) &= \frac{q_{21}}{q_{12} + q_{21}} + \left(p_0 - \frac{q_{21}}{q_{12} + q_{21}} \right) e^{-(q_{21}+q_{12})t}, \\ \varphi_2(t) &= \frac{q_{12}}{q_{12} + q_{21}} - \left(p_0 - \frac{q_{21}}{q_{12} + q_{21}} \right) e^{-(q_{21}+q_{12})t}. \end{aligned} \tag{14}$$

The time-varying weights $\varphi_1(t)$ and $\varphi_2(t)$ for the two regimes thus converge to their limiting probabilities, $\frac{q_{21}}{q_{21}+q_{12}}$ and $\frac{q_{12}}{q_{21}+q_{21}}$, respectively, and the transient part governed by the second eigenvalue λ_2, covers the relaxation from an initial distribution $(p_0, 1 - p_0)$ to the limiting distribution. In order to arrive at a closed-form solution for $u(x, t)$ and $w(x, t)$, one just has to multiply $\varphi_1(t)$ and $\varphi_2(t)$ to the solutions of the partial differential equations for $\theta_1(x, t)$and $\theta_2(x, t)$, i.e. two Brownian motions:

$$\theta_1 = \frac{1}{\sqrt{2\pi}\sigma_1 t} e^{-\frac{1}{2}(\frac{x-x_0}{\sigma_1 t})^2}, \theta_2 = \frac{1}{\sqrt{2\pi}\sigma_2 t} e^{-\frac{1}{2}(\frac{x-x_0}{\sigma_2 t})^2} \tag{15}$$

where x_0 is the observed realisation of the process at $t = 0$.

The form of the solution of the conditional density seems intuitively plausible: It combines the densities of both regime-specific stochastic differential equations with time-varying weights that can be interpreted as their probabilities of occurrence at time t. Over time, these weights converge to the limiting probabilities of both regimes as the influence of the initial condition gets lost over time. The unconditional transient density of the entire process is then given by the sum of both components: $\varphi_1(t)\theta_1(x, t) + \varphi_2(t)\theta_2(x, t)$.

Inspection shows that our *ansatz* can be generalized along many dimensions and could provide closed-form solutions for a wide array of regime-switching diffusions. First, mere inspection of eqs. (7) to (10) shows that the same approach could be applied to dynamic systems with any number of states. An example with a large number of states will be given in the next section. Second, we could allow for regime-dependent drift functions (as in eq. (1)) without having to modify the approach above: By simple inspection, one recognizes that the drifts would only enter in eqs. (9) which would, then, again just reproduce the regime-specific Fokker-Planck equations. $\theta_1(x,t)$ and $\theta_2(x,t)$ would correspond to the closed-form solutions of the pertinent SDEs governing the two regimes, and $\varphi_1(t)$ and $\varphi_2(t)$ would be determined as in the case without drift. As long as the density for all regime-specific processes are known (which might be different from each other) the overall process could also be described by a density in closed form. Third, multivariate extensions would also be straightforward (one just would have to replace the scalar-valued x_t by a vector in the above derivations) and availability of a closed-form solution would only hinge on availability of such solutions for all individual regimes. One could also think of applying this approach for transition rates depending on time and state, but the solvability of such more general cases would depend on the precise nature of the state-dependency.

3 An exact solution for a model with a large number of regimes: The Poisson multifractal model

In the following, we will apply the approach outlined above to a recently proposed model that has been formulated originally as a compound Brownian motion subordinate to a stochastic trading time defined by the cumulative density of a certain multifractal measure. We will show that this process can be cast into the framework of regime-switching diffusions, and provide the closed-form solution of its density for an arbitrary number of regimes.

Calvet and Fisher (2001) have introduced a so-called Poisson multifractal measure as a product of independent random variables with exponential density of their renewal times. When a new arrival occurs, the previous value of the pertinent component is replaced by a new draw from its underlying distribution. Here we assume draws from identical Binomial distributions

with the two realizations $M_t^i \in \{m_0, 2 - m_0\}$ which guarantees $E[M_t^i] = 1 \; \forall \; i, t$ with i the rank of the pertinent component $i = 1, \ldots, k$ and t the time index. The overall compound process is described by the mass $\prod_{i=1}^{k} M_t^i$ which is interpreted as a Markov model of stochastic trading time in an asset-pricing context. In the resulting compound asset-pricing model, the log-returns $x(t) = \ln P(t) - \ln P(0)$ are assumed to follow a Brownian motion subordinate to a time transformation $\theta(t)$:

$$x(t) = B[\theta(t)] \tag{16}$$

with $B[\cdot]$ the Brownian process and $\theta(t)$ stochastic trading time defined as the cumulative density of the Poisson multifractal measure as defined above.

Following the approach of previous multifractal processes adapted from statistical physics (Mandelbrot, 1974; Calvet *et al.*, 1997), Calvet and Fisher (2001) initially defined the Poisson multifractal on a finite interval $[0, T]$. However, this limitation appears unnecessary, as the process could easily be prolonged beyond "time" T without any disruption of its stochastic structure when the initial cascade process is completed. Hence, due to its Markov structure, the Poisson multifractal might be easily generalized to a continuous-time process for all $t \geq 0$. This feature distinguishes this type of multifractal model from its predecessor, the combinatorial Multifractal Model of Asset Returns of Calvet *et al.* (1997), and Calvet and Fisher (2002).

We could, then, alternatively formalize the Poisson multifractal with a discrete distribution of multipliers as a continuous-time stochastic process with regime switching. With Binomial distribution of the multipliers and k states, the realization of the product of these multipliers, $M_t = \prod_{i=1}^{k} M_t^i$, has 2^k possible values. Since the current draw of M_t defines the local volatility of a Brownian process via (16) as long as no arrival occurs in any of the Poissonian components, we can reformulate the compound process (16) as a diffusion process switching between 2^k states identified by their respective diffusion terms:

$$dx_t = \sigma(s_t) dW_t \tag{17}$$

with $\sigma(\cdot) = \sigma_i$ if $s_t = i$ $(i = 1, \ldots, 2^k)$, and s_t a continuous-time Markov chain governed by the infinitesimal matrix Q of intensity parameters q_{ij}.

The structure of the multifractal model generates a particular, sparse structure of the intensity matrix. Starting with the simplest of cases, for $k = 1$, the intensity matrix is that of a baseline regime-switching model with two states and identical probabilities to switch to the other state. Hence, $Q_{k=1}$ would assume the simple structure:

$$Q_{k=1} = \begin{pmatrix} -q_1 & q_1 \\ q_1 & -q_1 \end{pmatrix}. \tag{18}$$

For $k = 2$, $M_t = \prod_{i=1}^{2} M_t^i$ consists of two components with arrival intensities denoted, say, by q_1 and q_2 (the multifractal structure would imply a higher mean survival time of the second component, so q_2 would be smaller than q_1). If we arrange the four possible states as follows: $m_0 m_0$, $(1-m_0)m_0$, $m_0(1-m_0)$, $(1-m_0)(1-m_0)$, the intensity matrix would become:

$$Q_{k=2} = \begin{pmatrix} -(q_1+q_2) & q_1 & q_2 & 0 \\ q_1 & -(q_1+q_2) & 0 & q_2 \\ q_2 & 0 & -(q_1+q_2) & q_1 \\ 0 & q_2 & q_1 & -(q_1+q_2) \end{pmatrix}. \tag{19}$$

Similarly, for $k = 3$, we could choose the order of the eight states as $m_0 m_0 m_0$, $(1-m_0)m_0 m_0$, $m_0(1-m_0)m_0$, $(1-m_0)(1-m_0)m_0$, $m_0 m_0(1-m_0)$, $(1-m_0)m_0(1-m_0)$, $m_0(1-m_0)(1-m_0)$, $(1-m_0)(1-m_0)(1-m_0)$, with pertinent intensity matrix:

$$Q_{k=3} = \begin{pmatrix} -\beta & q_1 & q_2 & 0 & q_3 & 0 & 0 & 0 \\ q_1 & -\beta & 0 & q_2 & 0 & q_3 & 0 & 0 \\ q_2 & 0 & -\beta & q_1 & 0 & 0 & q_3 & 0 \\ 0 & q_2 & q_1 & -\beta & 0 & 0 & 0 & q_3 \\ q_3 & 0 & 0 & 0 & -\beta & q_1 & q_2 & 0 \\ 0 & q_3 & 0 & 0 & q_1 & -\beta & 0 & q_2 \\ 0 & 0 & q_3 & 0 & q_2 & 0 & -\beta & q_1 \\ 0 & 0 & 0 & q_3 & 0 & q_2 & q_1 & -\beta \end{pmatrix}, \tag{20}$$

with $\beta = q_1 + q_2 + q_3$.

If we continue like this, i.e. for each subsequent step of the cascade ordering the new set of states according to the previous sequence with first realization m_0 and then $1 - m_0$ added for the next step, we obtain a succession of quadratic block-diagonal matrices

$$Q_k = \begin{pmatrix} A_k & B_k \\ B_k & A_k \end{pmatrix} \tag{21}$$

in which block A_k is almost identical to the full matrix at the previous step, Q_{k-1}.

The Fokker-Planck equation for the probability density associated to such a process consists of a vector of 2^k state-wise partial differential equations. For instance, for $k = 2$ it is given by:

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= \sigma_1 \frac{\partial^2 u_1}{\partial x^2} - (q_1 + q_2)u_1 + q_1 u_2 + q_1 u_3 \\
\frac{\partial u_2}{\partial t} &= \sigma_2 \frac{\partial^2 u_2}{\partial x^2} + q_1 u_1 - (q_1 + q_2)u_2 + q_2 u_4 \\
\frac{\partial u_3}{\partial t} &= \sigma_3 \frac{\partial^2 u_3}{\partial x^2} + q_2 u_1 - (q_1 + q_2)u_3 + q_1 u_4 \\
\frac{\partial u_4}{\partial t} &= \sigma_4 \frac{\partial^2 u_4}{\partial x^2} + q_2 u_2 + q_1 u_3 - (q_1 + q_2)u_4
\end{aligned} \tag{22}$$

Here, we have defined the state-wise densities $u_i(x,t) = f(x, s = i; t)$ for $i = 1, ..., 4$.

Although the dimension of the intensity matrix can become arbitrarily large with increasing k, we can obtain a general closed form solution for the vector of partial differential equations (22) and its generalizations to higher k. We again use the *ansatz* of eq. (7) as a starting point and decompose the state-dependent densities as follows:

$$u_i(x,t) = \varphi_i(t)\theta_i(x,t) \tag{23}$$

Upon substitution, we get a similar system of equations as for the simple case of a two-state regime switching diffusion depicted in eq. (5). A solution of this system (for any k) consists analogously in $\theta_i(x,t)$ solving the 2^k independent partial differential equations:

$$\frac{\partial \theta_i}{\partial t} = \frac{1}{2}\sigma_i \frac{\partial^2 \theta_i}{\partial x^2}, \tag{24}$$

and the $\varphi_i(t)$ being solutions to a system of 2^k ordinary linear differential equations for the flow of probability between states. We will show in the following that we can determine analytically the full set of eigenvalues and eigenvectors of this system, and, therefore, obtain a closed-form solution for the transient density of a Poisson multifractal asset-pricing model for any k. To this end, we will compute the full set of eigenvalues, their eigenvectors, and integration constants depending on state probabilities at time 0.

Eigenvalues

To set the scene, we start again with $k = 1$ leading to the following variation of the system of equations (10):

$$\begin{aligned} \varphi_1' \theta_1 &= -q_1 \varphi_1 \theta_1 + q_1 \varphi_2 \theta_2 \\ \varphi_2' \theta_2 &= q_1 \varphi_1 \theta_1 - q_1 \varphi_2 \theta_2. \end{aligned} \tag{25}$$

We denote by $\lambda^{(k)}$ the vector of eigenvalues for a Poisson model with k hierarchical components. The solutions $\varphi_1(t) = \alpha_1 e^{\lambda_1^{(1)} t}$, $\varphi_2(t) = \alpha_2 e^{\lambda_2^{(1)} t}$ of eqs (25) are characterized by eigenvalues that satisfy:

$$\det(D_1) = \det \begin{pmatrix} -q_1 - \lambda^{(1)} & q_1 \\ q_1 & -q_1 - \lambda^{(1)} \end{pmatrix} = 0, \tag{26}$$

which leads to $\lambda_1^{(1)} = 0$ and $\lambda_2^{(1)} = -2q_1$.

Now move on to $k = 2$. For simplification of notation, we denote by $\rho^{(k)} = \sum_{j=1}^{k} q_j + \lambda^{(k)}$ (which here is $q_1 + q_2 + \lambda^{(2)}$) and multiply D_2 by -1. Hence, the eigenvalues are determined by

$$\det(D_2) = \det \begin{pmatrix} \rho^{(2)} & -q_1 & -q_2 & 0 \\ -q_1 & \rho^{(2)} & 0 & -q_2 \\ -q_2 & 0 & \rho^{(2)} & -q_1 \\ 0 & -q_2 & -q_1 & \rho^{(2)} \end{pmatrix}. \tag{27}$$

Direct solution leads to $\rho_1^{(2)} = q_1 + q_2$, $\rho_2^{(2)} = q_1 - q_2$, $\rho_3^{(2)} = q_2 - q_1$, $\rho_4^{(2)} = -(q_1 + q_2)$. Therefore, we obtain $\lambda_1^{(2)} = 0$, $\lambda_2^{(2)} = -2q_2$, $\lambda_3^{(2)} = -2q_1$, $\lambda_4^{(2)} = -2(q_1 + q_2)$.

To exploit the systematic patterns in the construction of the intensity matrices, note that

$$D_2 = \begin{pmatrix} A_2 & B_2 \\ B_2 & A_2 \end{pmatrix} \text{ with } A_2 = \begin{pmatrix} \rho^{(2)} & -q_1 \\ -q_1 & \rho^{(2)} \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} -q_2 & 0 \\ 0 & -q_2 \end{pmatrix}.$$

For symmetric, block-diagonal matrices, it holds that (cf. Gantmacher, 1974, c.II)

$$\det(D_2) = \det(A_2)\det(A_2 - B_2 A_2^{-1} B_2). \tag{28}$$

Now, D_2, A_2, and B_2 are all symmetric matrices. Recall that two matrices R and S are called similar if there exists an invertible matrix P such that $P^{-1}SP = R$. Clearly, such a matrix exists for the similarity transformation between

$$A_2 = \begin{pmatrix} \rho^{(2)} & -q_1 \\ -q_1 & \rho^{(2)} \end{pmatrix} \text{ and the diagonal matrix } \Lambda_2 = \begin{pmatrix} \rho^{(2)} - q_1 & 0 \\ 0 & \rho^{(2)} + q_1 \end{pmatrix}.$$

Since similar matrices have the same eigenvalues, we can replace $\det(A_2)$ by $\det(\Lambda_2)$ in (28). In fact, we can also replace A_2 and its inverse by Λ_2 and its inverse in the second component of this product, $A_2 - B_2 A_2^{-1} B_2$. Namely, if P_2 is the matrix that fulfills:

$$\Lambda_2 = P_2^{-1} A_2 P_2, \tag{29}$$

then performing the same operation for the second term leads to:

$$\begin{aligned} P_2^{-1}\left(A_2 - B_2 A_2^{-1} B_2\right) P_2 &= P_2^{-1} A_2 P_2 - P_2^{-1} B_2 A_2^{-1} B_2 P_2 \\ &= \Lambda_2 - q_2^2 I P_2^{-1} A_2^{-1} P_2 \\ &= \Lambda_2 - q_2^2 I P_2^{T} A_2^{-1} P_2 \\ &= \Lambda_2 - q_2^2 I P_2 A_2^{-1} P_2^{-1} \\ &= \Lambda_2 - q_2^2 I \Lambda_2^{-1}, \end{aligned} \tag{30}$$

where I denotes the identity matrix. In this sequence of operations, the second line makes use of the simple diagonal structure of B_2 with identical values along its diagonal. The third line exploits the fact that A_2 and A_2^{-1} are symmetric matrices and are, therefore, orthogonally diagonalizable, so that $P_2^{-1} = P_2^T$. The last identity follows from this fact.

With this replacement, we arrive at

$$\begin{aligned}
\det(D_2) &= \det(\Lambda_2)\det(\Lambda_2 - q_2^2 I \Lambda_2^{-1}) \\
&= (\rho^{(2)} - q_1)(\rho^{(2)} + q_1)\left[\begin{pmatrix} \rho^{(2)} - q_1 & 0 \\ 0 & \rho^{(2)} + q_1 \end{pmatrix} - q_2^2 \begin{pmatrix} \frac{1}{\rho^{(2)} - q_1} & 0 \\ 0 & \frac{1}{\rho^{(2)} + q_1} \end{pmatrix}\right] \\
&= (\rho^{(2)} - q_1)(\rho^{(2)} + q_1)\left[\left(\rho^{(2)} - q_1 - q_2^2 \frac{1}{\rho^{(2)} - q_1}\right)\left(\rho^{(2)} + q_1 - q_2^2 \frac{1}{\rho^{(2)} + q_1}\right)\right] \\
&= \left[(\rho^{(2)} - q_1)^2 - q_2^2\right]\left[(\rho^{(2)} + q_1)^2 - q_2^2\right].
\end{aligned} \tag{31}$$

Solving the two equations we get:

$$\rho^{(2)^2} - 2q_1\rho^{(2)} + q_1^2 - q_2^2 = 0 \tag{32}$$

leading to $\rho_{1/2}^{(2)} = q_1 \pm q_2$, and

$$\rho^{(2)^2} + 2q_1\rho^{(2)} + q_1^2 - q_2^2 = 0 \tag{33}$$

leading to $\rho_{3/4}^{(2)} = -q_1 \pm q_2$.

Since $\lambda_i^{(2)} = \rho_i^{(2)} - q_1 - q_2$, the four eigenvalues of the original system are: 0, $-2q_1$, $-2q_2$, and $-2(q_1 + q_2)$ as we had already determined via direct solution. The advantage of the second approach to this hitherto simple problem is that it can be used to show how the set of eigenvalues develops from stage to stage for a process of arbitrary many cascade levels k.

Moving to $k = 3$ it is obvious that due to the symmetry of D_3, A_3 and B_3, it holds again that

$$\det(D_3) = \det(\Lambda_3)\det(\Lambda_3 - q_3^2 I \Lambda_3^{-1}) \tag{34}$$

where Λ_3 is the diagonal matrix with entries $\rho^{(3)} - q_1 \pm q_2$ and $\rho^{(3)} + q_1 \pm q_2$. We, therefore get:

$$\begin{aligned}\det(D_3) = ((\rho^{(3)} - (q_1 + q_2))^2 - q_3^2)((\rho^{(3)} - (q_1 - q_2))^2 - q_3^2) \\ ((\rho^{(3)} - (q_2 - q_1))^2 - q_3^2)((\rho^{(3)} + (q_1 + q_2))^2 - q_3^2)\end{aligned} \tag{35}$$

For each of the four terms, we get two new solutions, say $\rho^{(3)}_{i,+}$ and $\rho^{(3)}_{i,-}$ $(i = 1, ..., 4)$ which can be written in dependency on the "previous" ones as $\rho^{(3)}_{i,\pm} = \rho^{(2)}_i \pm q_3$, and, since $\lambda^{(3)}_i = \rho^{(3)}_i - q_1 - q_2 - q_3$ (recall that the definition of $\lambda^{(k)}$ changes from step to step in that the new parameter q_k is subtracted on the right-hand side) the new $\lambda^{(3)}_i$ emerge as $\lambda^{(3)}_{i,1} = \lambda^{(2)}_i - 2q_3$ and $\lambda^{(3)}_{i,2} = \lambda^{(2)}_{i,1}$.

Now moving on to the next stage, $k = 4$, we find the eight solutions of $\rho^{(3)}_i$, namely $\{q_1+q_2+q_3,\ q_1+q_2-q_3,\ q_1-q_2+q_3,\ q_1-q_2-q_3,\ q_2-q_1+q_3,\ q_2-q_1-q_3,\ -q_1-q_2+q_3,\ -q_1-q_2-q_3\}$ appear in the diagonal of Λ_4. It holds that $\det(D_4) = \det(\Lambda_4)\det(\Lambda_4 - q_4^2 I \Lambda_4^{-1})$, which can be compactly written:

$$\det(D_4) = \prod_{i=1}^{8}\left[\left(\rho^{(4)} - \rho^{(3)}_i\right)^2 - q_4^2\right] \tag{36}$$

Every $\rho^{(3)}_i$ bifurcates into two $\rho^{(4)}_{i,\pm} \pm q_4$, and every eigenvalue $\lambda^{(3)}_i$ bifurcates into two

$$\lambda^{(4)}_{i,1,2} = \lambda^{(3)}_i + \begin{cases} -2q_4 \\ \\ 0. \end{cases} \tag{37}$$

Following up this development we see that the 2^k eigenvalues for any arbitrary number k of cascade steps are given by all permutations of

$$\sum_{i=1}^{k} -2q_i I_i, \qquad \text{with} \quad I_i = \{0, 1\}. \tag{38}$$

Eigenvectors:

Now, let us determine the eigenvectors α_i associated to the eigenvalues $\lambda^{(k)}_i$. Again, we can obtain a closed-form solution, and we try to develop some intuition by first considering cases

with small k. For $k = 2$, the four eigenvalues are $\lambda_i^{(2)} = \{0, -2q_1, -2q_2, -2(q_1 + q_2)\}$. The pertinent eigenvectors are determined by the following systems[2]:

$$\begin{pmatrix} -\delta & q_1 & q_2 & 0 \\ q_1 & -\delta & 0 & q_2 \\ q_2 & 0 & -\delta & q_1 \\ 0 & q_2 & q_1 & -\delta \end{pmatrix} \begin{pmatrix} \alpha_i^{(1)} \\ \alpha_i^{(2)} \\ \alpha_i^{(3)} \\ \alpha_i^{(4)} \end{pmatrix} = 0, \tag{39}$$

with $\delta = q_1 + q_2 + \lambda_i^{(2)}$, and $\alpha_i^{(j)}$ the jth entry of eigenvector i.

One easily confirms that the eigenvectors for the four eigenvalues above -in their order of appearance- are given by:

$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \text{and} \quad \alpha_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

The first one for $\lambda_1^{(2)} = 0$ is obvious as the vector of ones just makes each equation on the left-hand side vanish. $\lambda_2^{(2)} = -2q_1$ leads to diagonal elements $q_1 - q_2$ in the coefficient matrix. In order for every single equation to turn out zero, this now requires neighboring elements of α_2 to have opposite sign. Similarly, for $\lambda_3^{(2)} = -2q_2$, the diagonal entries become $q_2 - q_1$. Zero is obtained if the first and second dyadic pair have different signs. Finally, for $\lambda_4^{(2)} = -2(q_1 + q_2)$, both each of the two entries of a dyadic pair as well as the two pairs have to have different signs. Note that this would be achieved by first multiplying every second element ($\alpha_4^{(2)}$ and $\alpha_4^{(4)}$) by -1, and subsequently multiplying the second dyadic pair ($\alpha_4^{(3)}$ and $\alpha_4^{(4)}$) by -1.

A glance at $k = 3$ suffices to show that due to the block structures of the model, this is a general principle of construction. Eigenvectors for $k = 3$ are obtained via:

[2]To ease notation, we do not add an index for the hierarchical level k for eigenvectors.

$$\begin{pmatrix} -\varrho & q_1 & q_2 & 0 & q_3 & 0 & 0 & 0 \\ q_1 & -\varrho & 0 & q_2 & 0 & q_3 & 0 & 0 \\ q_2 & 0 & -\varrho & q_1 & 0 & 0 & q_3 & 0 \\ 0 & q_2 & q_1 & -\varrho & 0 & 0 & 0 & q_3 \\ q_3 & 0 & 0 & 0 & -\varrho & q_1 & q_2 & 0 \\ 0 & q_3 & 0 & 0 & q_1 & -\varrho & 0 & q_2 \\ 0 & 0 & q_3 & 0 & q_2 & 0 & -\varrho & q_1 \\ 0 & 0 & 0 & q_3 & 0 & q_2 & q_1 & -\varrho \end{pmatrix} \begin{pmatrix} \alpha_i^{(1)} \\ \alpha_i^{(2)} \\ \alpha_i^{(3)} \\ \alpha_i^{(4)} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_i^{(8)} \end{pmatrix} = 0 \tag{40}$$

with $\varrho = q_1 + q_2 + q_3 + \lambda_i^{(3)}$.

We quickly convince ourselves that eigenvalues and eigenvectors are as follows:

$$\begin{aligned}
\lambda_1^{(3)} &= 0 && \text{with} \quad \alpha_1 &= (1,1,1,1,1,1,1,1)' \\
\lambda_2^{(3)} &= -2q_1 && \text{with} \quad \alpha_2 &= (1,-1,1,-1,1,-1,1,-1)' \\
\lambda_3^{(3)} &= -2q_2 && \text{with} \quad \alpha_3 &= (1,1,-1,-1,1,1,-1,-1)' \\
\lambda_4^{(3)} &= -2q_3 && \text{with} \quad \alpha_4 &= (1,1,1,1,-1,-1,-1,-1)' \\
\lambda_5^{(3)} &= -2(q_1+q_2) && \text{with} \quad \alpha_5 &= (1,-1,-1,1,1,-1,-1,1)' \\
\lambda_6^{(3)} &= -2(q_1+q_3) && \text{with} \quad \alpha_6 &= (1,-1,1,-1,-1,1,-1,1)' \\
\lambda_7^{(3)} &= -2(q_2+q_3) && \text{with} \quad \alpha_7 &= (1,1,-1,-1,-1,-1,1,1)' \\
\lambda_8^{(3)} &= -2(q_1+q_2+q_3) && \text{with} \quad \alpha_8 &= (1,-1,-1,1,-1,1,1,-1)'.
\end{aligned}$$

With higher order addends for $k > 2$ the principle of construction remains the same. We, therefore, obtain the eigenvector associated to eigenvalue $\lambda_i^{(k)}$ as a vector consisting of -1's and 1's. Starting with a unitary vector of length 2^k, we perform a series of multiplications, in which its elements $2^i - 2^{i-1} + 1 : 2^i$, $2 \cdot 2^i - 2^{i-1} + 1 : 2 \cdot 2^i, \ldots, 2^{k-i} \cdot 2^i - 2^{i-1} + 1 : 2^k$ are multiplied by -1 for all i that have non-zero values in the indicator function I_i of eq. (38) for the pertinent eigenvalue.

Integration constants for given initial conditions:

Finally, we determine the constants $A_1, \ldots, A_n$ in dependence on given initial conditions, the probabilities of states $1, \ldots, n$ $(n = 2^k)$ at time $t = 0$. The later are denoted by

$P_1(0), P_2(0), \ldots, P_n(0)$ with $\sum_{i=1}^{n} P_i(0) = 1$.

For arbitrary k, at time zero, the generalization of system (13) boils down to

$$\begin{array}{rcccl} \varphi_1(0) & = & P_1(0) & = & A_1\alpha_1^{(1)} + A_2\alpha_2^{(1)} + A_3\alpha_3^{(1)} + \cdots + A_n\alpha_n^{(1)} \\ \varphi_2(0) & = & P_2(0) & = & A_1\alpha_1^{(2)} + A_2\alpha_2^{(2)} + A_3\alpha_3^{(2)} + \cdots + A_n\alpha_n^{(2)} \\ \vdots & & & & \\ \varphi_n(0) & = & P_n(0) & = & A_1\alpha_1^{(n)} + A_2\alpha_2^{(n)} + A_3\alpha_3^{(n)} + \cdots + A_n\alpha_n^{(n)}. \end{array} \tag{41}$$

In matrix notation, this is compactly expressed as:

$$P(0) = \alpha \cdot A, \tag{42}$$

with $P(0) = (P_1(0), P_2(0), \ldots, P_n(0))'$, $A = (A_1, A_2, \ldots, A_n)'$, and α the matrix made up by the eigenvectors $\alpha = [\alpha_i^{(j)}]$. Solving for the vector of constants leads to:

$$A = \alpha^{-1} P(0). \tag{43}$$

This requires to determine the inverse of α. Again, we can determine the structure of α^{-1} for any k which enables us to directly implement also this part of the solution without having to incur the cost of solving large systems of equations over and over again in empirical applications.

We again inspect the simplest cases which easily reveal the general pattern. For developing intuition, we start with $k = 1$. In this case, we easily derive that the inverse of

$$\alpha = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ is } \alpha^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \tag{44}$$

This leads to:

$$A_1 = 0.5P_1(0) + 0.5P_2(0) = 0.5P_1(0) + 0.5(1 - P_1(0)) = 0.5,$$

$$A_2 = 0.5P_1(0) - 0.5(1 - P_1(0)) = P_1(0) - 0.5.$$

Hence, for given initial conditions, we arrive at the completely determined solution for the

time-dependent probability of both states:

$$\begin{aligned} \varphi_1(t) &= 0.5 + (P_1(0) - 0.5)e^{-2q_1 t}, \\ \varphi_2(t) &= 0.5 - (P_1(0) - 0.5)e^{-2q_1 t}. \end{aligned} \tag{45}$$

As in the more general case of two regimes with arbitrary intensities for the switching process, the first term, 0.5, is the limiting probability of each state whereas the second term is the transient contribution due to initial probabilities that deviate from the limiting ones.

Now we move on to $k = 2$. Arranging the eigenvalues and eigenvectors as in (44), we have:

$$\alpha = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \text{ with inverse } \alpha^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}. \tag{46}$$

The pattern of the entries α^{-1} suggests that the positive and negative entries of α are mirrored in positive and negative entries of α^{-1}. To see that this is indeed the case, define the entries of α^{-1} as $\gamma = [\gamma_{ij}]$. We then obtain for the first column:

$$\begin{aligned} \gamma_{11} + \gamma_{21} + \gamma_{31} + \gamma_{41} &= 1, \\ \gamma_{11} - \gamma_{21} + \gamma_{31} - \gamma_{41} &= 0, \\ \gamma_{11} + \gamma_{21} - \gamma_{31} - \gamma_{41} &= 0, \\ \gamma_{11} - \gamma_{21} - \gamma_{31} + \gamma_{41} &= 0. \end{aligned} \tag{47}$$

Clearly, this leads to $\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = 0.25$. Moving to large k, we similarly obtain one of the 2^k equations being

$$\sum_{j=1}^{2^k} \gamma_{j1} = 1 \tag{48}$$

and the remaining $2^k - 1$ equations all having as many positive as negative entries on the left-hand side and zero on the right-hand side. It follows that the first column of γ for any k is

occupied by uniform entries $\frac{1}{2^k}$.

Now turn to the second column for $k = 2$ which is determined by:

$$\begin{aligned} \gamma_{12} + \gamma_{22} + \gamma_{32} + \gamma_{42} &= 0, \\ \gamma_{12} - \gamma_{22} + \gamma_{32} - \gamma_{42} &= 1, \\ \gamma_{12} + \gamma_{22} - \gamma_{32} - \gamma_{42} &= 0, \\ \gamma_{12} - \gamma_{22} - \gamma_{32} + \gamma_{42} &= 0. \end{aligned} \tag{49}$$

leading to $\gamma_{12} = \gamma_{32} = 0.25$, $\gamma_{22} = \gamma_{42} = -0.25$. Hence, the γ_{ij} are all 0.25 in absolute value and share the same sign, with the pertinent entries of α. Similar structures apply for the third and forth columns leading to the pattens of eq. (46). Does this pattern also extend to higher-dimensional cascades? First, in every column, the element on the main diagonal will consist of an equation of the format $\sum_j^{2^k} \gamma_{jl} = 1$ in which half of all entries are positive and the other half are negative. Second, the remaining $2^k - 1$ equations can be obtained from the one for the diagonal entry by at least one of the multiplicative change of sign operations that have been applied to obtain the eigenvectors composed of an equal number of $+1'$s and $-1'$s (except for the first eigenvector). This will change the sign of one entry of each dyadic pair, of two out of a group of four (in the natural order), four out of eight summands etc, and will, therefore always lead to pairwise cancelation of the terms that in the equation for the diagonal entry have summed up to one. Hence, the inverse of α is a matrix with entries of absolute value equal to 2^{-k} and signs identical to those of the respective entries of α.

To complete the example of $k = 2$, we find, therefore, that:

$$\begin{aligned} A_1 &= 0.25, A_2 = 0.25(P_1(0) + P_3(0) - P_2(0) - P_4(0)), \\ A_3 &= 0.25(P_1(0) + P_2(0) - P_3(0) - P_4(0)), \\ A_4 &= 0.25(P_1(0) - P_2(0) - P_3(0) + P_4(0)). \end{aligned}$$

Inserting this together with the results for eigenvalues and eigenvectors into the equivalent of eq. (12), we arrive at the fully specified trajectories of the state probabilities conditional on the boundary (initial) condition $P(0)$. Note that no numerical operations are involved at any

point: the values of $\varphi_1(t), \ldots, \varphi_n(t)$ can just be obtained from implementing the formula for any requested value of $t \geq 0$. Together with the solutions of $\theta_1(x,t), \ldots, \theta_n(x,t)$ that also have the format of eq. (15), i.e.

$$\theta_i(x,t) = \frac{1}{\sqrt{2\pi}\sigma_i t} e^{-\frac{1}{2}(\frac{x-x_0}{\sigma_i t})^2} \tag{50}$$

we obtain the shape of the transient density at any time t.

4 An illustration: Maximum likelihood estimation of the parameters of the Poisson multifractal model

Closed-form solutions of continuous-time models facilitate a number of important tasks: First, they enable the researcher to use straightforward maximum likelihood methods for estimation of the parameters of a hypothesized continuous-time data generating process in any application of such processes. Second, in particular in financial economics, the pricing of derivative assets depends on the data-generating process of the underlying and its parameters, and closed form transitional densities for all possible expiration dates would be useful in the determination of arbitrage-free derivative prices and related quantities. In short, having a closed-form solution at one's disposal, makes a large spectrum of often computation-intense approximation methods obsolete in estimation and financial engineering. We leave the later aspect for future research, and here confine ourselves to an illustration of straightforward ML estimation of the parameters of the Poisson multifractal model.

Having typically data at our disposal at (equi-) distant intervals $t = i \triangle t$, $i = 0, ..., T$, due to the Markov property of the process, the log-likelihood assumes the form:

$$l(\theta) = \frac{1}{T} \sum_{t=1}^{T} ln(f(x_t | I_{t-1}, \theta) \tag{51}$$

where I_t is information at time t and θ is the vector of parameters including the regime-dependent parameters of the drift (absent in our case) and diffusion functions as well as the parameters governing the intensity matrix. Note that our derivation of the closed-form solutions

for the densities implies that the probability of a change of regime *at any point in time between* two neighbouring observations has been exactly taken into account. Previous approximative solutions to the likelihood function (51) have instead often used an analogue of the so-called discrete-time Hamilton algorithm by decomposing the likelihood into their regime-specific parts and discrete transition probabilities between states over the sampling interval. The parameters of the conditional densities of the different regimes are then either assumed to depend on the regime at time t of $t-1$. Such a discretization will be inferior to the exact MC approach and would be the more cumbersome the larger the discrete time steps between observations.

Our illustration of the ML approach based on closed form densities considers Binomial Poisson multifractals with different numbers of hierarchical components ($k = 2, 4, 8$) and Binomial distributions $\{m_0, 2 - m_0\}$ with m_0 assuming values $m_0 = 1.25, 1.5$ and 1.75. The switching intensities are characterized by a geometric progression which we formalize by $q_i = \lambda b^{i-1}, i = 1, ..., k$. To normalize each process in a way to have an intensity of 1 for arrivals at the component with highest frequency, we set $\lambda = b^{1-k}$, and we let $b = 2$ which yields a progression of switching frequency between states by a factor two. The following Monte Carlo exercise uses an exact simulation algorithm inspired by the algorithm proposed by Glasserman (2004, s.3) for simple jump processes, so that there is no discretization error in the Monte Carlo samples (the algorithm is available in C code upon request). Estimation is performed with sample lengths of $T = 5,000$ and $T = 10,000$ entries.

Table 1 shows the results. As one can see, the performance of the estimation improves when moving from the upper left-hand side of the table to the lower right-hand side. I summarize here the major observations:

1. The binomial parameter m_0 is always estimated very precisely, without almost any bias and expected improvement for longer samples. Higher values of m_0 seem to facilitate estimation, and higher numbers of Poisson components have initially the same effect (from $k = 2$ to $k = 4$) while there is less difference in this respect between the specifications $k = 4$ and $k = 8$.

2. Estimation of b and λ is somewhat problematic at the upper left-hand end of the table, i.e. if m_0 is small, and the number of Poisson components is small as well. Low m_0 means little

difference between regimes, while a small k also implies that the switches between these regimes are relatively frequent. The estimation has, therefore, problems in distinguishing between the constant (λ) and the level-specific part (b) of the intensities. The large mean values for the estimates of b in the upper left corner are due to some outliers while the median value of unity shows that the estimation cannot properly distinguish in many cases between four regimes ($k = 2$) with mild heterogeneity and one regime only (which would be the borderline solution with $b = 1$).

However, both an increase of m_0 (leading to more pronounced heterogeneity) across regimes as well as an increase of k (leading to a more diverse spectrum of regimes) lead to better estimates for b and λ. Note also that in all cases the quality of all parameter estimates improves with the length of the sample showing that all parameters can, in principle, be identified due to the non-linearity of the model.[3] Note also that, if anything, processes with more regimes (the number of regimes is increasing in our specifications from $2^2 = 4$ through $2^4 = 16$ to $2^8 = 256$) have rather more precise estimates which is not what one would intuitively expect. This feature appears characteristic of multifractal models and it might be due to the fact that specifications with a higher number of regimes do not come along with a higher number of parameters. This feature provides for a high robustness of such processes against misspecification of the number of cascade levels and regimes.

5 Conclusion

This paper has shown how exact solutions for the transient densities of a relatively large class of continuous-time regime-switching diffusions can be obtained. While we have only considered univariate pure diffusions with Markov-switching regime-specific variances, mere inspection of our results shows that the same approach would also be applicable to a large number of processes with linear and non-linear drift functions, multivariate settings, and possibly also to certain cases with state-dependent intensities for switches between regimes. Since closed-form solutions facilitate estimation and applications of regime-switching processes, this availability

[3] Only for $k = 1$, would λ and b not be identified so that only their product could be estimated.

will release the applied researcher from having to resort to often computation-intensive approximation methods for various purposes.

We have also applied this solution method to the Poisson multifractal model proposed by Calvet and Fisher (2001) which allows for an arbitrarily high number of regimes. With a closed form solution to the density of this process, full maximum likelihood estimation becomes feasible which we have illustrated to work well with examples including up to 256 different volatility regimes (but we would not expect any deterioration even for larger numbers). As far as we can see, this is also the first approach towards parameter estimation for the continuous-time Poisson multifractal, while previous empirical applications have concentrated on the discretized counterpart of this model (Calvet and Fisher, 2004; Lux, 2008).

References

Aït-Sahalia, Y., (2007) *Estimating Continuous-Time Models Using Discretely Sampled Data.* In: Blundell, R., Persson, T., Newey, W. K., (Eds.) Advances in Economics and Econometrics, Theory and Applications, Econometric Society Monographs. Cambridge University Press, MA.

Barone-Adesi, G., Whaley, R. E. (1987) Efficient analytic approximation of American option values, *The Journal of Finance* 42(2), 301-320.

Calvet, L. E., Fearnley, M., Fisher, A. J. and M. Leippold (2013) *What's beneath the surface? Option pricing with multifrequency latent states.* Working paper, HEC Paris, University of British Columbia, and University of Zürich.

Calvet, L., and A. Fisher (2001) Forecasting multifractal volatility, *Journal of Econometrics*, 105, 27–58.

Calvet, L. E. and A. J. Fisher (2004) How to forecast long-run volatility: regime switching and the estimation of multifractal processes, *Journal of Financial Econometrics* 2(1), 49-83.

Calvet, L., A. Fisher and B. Mandelbrot (1997) A Multifractal Model of Asset Returns, *Cowles Foundation Discussion Papers*, nos. 1164–1166, Yale University.

Calvet, L. E., Fisher, A. J. and L. Wu (2013) *Staying on top of the curve: A cascade model of term structure dynamics.* Available at SSRN: http://dx.doi.org/10.2139/ssrn.1573392.

Chen, F., Diebold, F. X. and F. Schorfheide (2013) A Markov-switching multifractal inter-trade duration model, with application to US equities, *Journal of Econometrics*, in press.

Choi, S. (2009) Regime-switching univariate diffusion models of the short-term interest rate, *Studies in Nonlinear Dynamics and Econometrics* 13(1).

Dai, Q. and K. Singleton (2003) Term structure dynamics in theory and reality, *Review of Financial Studies* 16(3), 631-678.

Elliott, R. J., Siu, T. K. and A. Badescu (2011) On pricing and hedging options in regime-switching models with feedback effect, *Journal of Economic Dynamics and Control* 35(5), 694-713.

Elliott, R. J., Siu, T. K. and L. Chan (2007) Pricing volatility swaps under Heston's stochastic volatility model with regime switching, *Applied Mathematical Finance* 14(1), 41-62.

Elliott, R. J., Siu, T. K. and L. Chan (2008a) A PDE approach for risk measures for derivatives with regime switching, *Annals of Finance* 4(1), 55-74.

Elliott, R. J., Krishnamurthy, V., and Sass, J. (2008b). Moment based regression algorithms for drift and volatility estimation in continuous-time Markov-switching models. *The Econometrics Journal*, 11(2), 244-270.

Fuh, C. D., Ho, K. W. R., Hu, I. and R. H. Wang (2012) Option Pricing with Markov Switching, *Journal of Data Science* 10(3), 483-509.

Gantmacher, F. (1974) *The Theory of Matrices*, vol. I, Chelsea Publ. Co., New York.

Glasserman, P. (2004) *Monte Carlo Methods in Financial Engineering*. Springer, Berlin.

Guo, X. and Zhang, Q., (2004), Closed-form solutions for perpetual American put options with regime switching. *SIAM Journal on Applied Mathematics*, 64(6), 2034-2049.

Hahn, M., Frühwirth-Schnatter, S., and J. Sass (2010) Markov chain Monte Carlo methods for parameter estimation in multidimensional continuous time Markov switching models, *Journal of Financial Econometrics* 8(1), 88-121.

Hamilton, J. D. (1989) A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle, *Econometrica*, 57, 357-384.

Hamilton, J. D. (1994) *Time-Series Analysis*. Princeton University Press.

Kauschke, J. (2010) *Agent-Based Models for Financial Markets*, Ph.D. thesis. University of Kiel.

Mandelbrot, B. (1974) Intermittent turbulence in self-similar cascades: Divergence of high moments and dimension of the carrier, *Journal of Fluid Dynamics*, 62, 331–358.

Mao, X. and Yuan, C., (2006), *Stochastic differential equations with Markovian switching.* Imperial College Pr.

Mao, X. et al. (2005) Numerical methods for stationary distributions of stochastic differential equations with Markovian switching, *Journal of Computational and Applied Mathematics* 174, 1-27.

Metzner, P., Horenko, I. and C. Schütte (2007) Generator estimation of Markov jump processes on incomplete observations non-equidistant in time, *Physical Review E*, 76, 066702.

Liechty, J. C. and G. O. Roberts (2001) Markov chain Monte Carlo methods for switching diffusion models, *Biometrika* 88(2), 299-315.

Lux, T. (2008) The markov-switching multifractal model of asset returns, *Journal of Business and Economic Statistics* 26(2), 194-210.

Lux, T. and T. Kaizoji (2007) Forecasting volatility and volume in the Tokyo stock market: Long memory, fractality and regime switching, *Journal of Economic Dynamics and Control* 31(6), 1808-1843.

Polyanin, A. and A. Manzhirov (2007), *Handbook of Mathematics for Engineers And Scientists*, London, Chapman & Hall.

Ramponi, A. (2011) Mixture dynamics and regime switching diffusions with application to option pricing, *Methodology and Computing in Applied Probability* 13, 349-368.

Saporta, B., and Yao, J. F. (2005), Tail of a linear diffusion with Markov switching. *The Annals of Applied Probability*, 15, 992-1018.

Smith, D. R. (2002) Markov-switching and stochastic volatility diffusion models of short-term interest rates. *Journal of Business and Economic Statistics* 20(2), 183-197.

Skorokhod, A. V. (2009) *Asymptotic Methods in the Theory of Sochastic Differential Equations.* Providence, American Mathematical Society.

Yin, G., and Zhu, C. (2010) *Hybrid Switching Diffusions: Properties and Applications.* New York, Springer.

Table 1: ML Parameter Estimates

						k = 2									
T	par.	value	mean	median	FSSE	RMSE	value	mean	median	FSSE	RMSE	value	mean	median	FSSE	RMSE
	λ	0.5	2.006	0.779	3.764	4.053	0.5	0.799	0.645	0.731	0.790	0.5	0.569	0.546	0.228	0.238
5000	b	2	1.723	1.000	1.791	1.812	2	2.149	1.000	1.997	2.002	2	2.450	1.832	1.807	1.861
	m_0	1.25	1.244	1.250	0.069	0.070	1.5	1.500	1.499	0.036	0.036	1.75	1.749	1.750	0.019	0.019
	λ	0.5	1.004	0.704	1.314	1.407	0.5	0.684	0.616	0.313	0.363	0.5	0.551	0.511	0.156	0.164
10000	b	2	2.215	1.000	2.104	2.114	2	2.121	1.486	1.543	1.547	2	2.094	2.027	0.939	0.943
	m_0	1.25	1.249	1.251	0.029	0.029	1.5	1.500	1.501	0.017	0.017	1.75	1.750	1.750	0.009	0.009
						k = 4									
T	par.	value	mean	median	FSSE	RMSE	value	mean	median	FSSE	RMSE	value	mean	median	FSSE	RMSE
	λ	0.125	0.256	0.192	0.229	0.264	0.125	0.160	0.129	0.102	0.108	0.125	0.149	0.129	0.080	0.083
5000	b	2	1.966	1.000	1.557	1.557	2	2.190	2.019	1.086	1.102	2	2.025	1.983	0.583	0.584
	m_0	1.25	1.253	1.255	0.036	0.036	1.5	1.499	1.500	0.026	0.026	1.75	1.749	1.749	0.015	0.015
	λ	0.125	0.173	0.143	0.099	0.110	0.125	0.133	0.127	0.037	0.038	0.125	0.129	0.125	0.025	0.026
10000	b	2	1.975	1.837	0.965	0.965	2	2.007	1.979	0.355	0.355	2	2.000	2.001	0.218	0.218
	m_0	1.25	1.251	1.251	0.016	0.016	1.25	1.499	1.500	0.011	0.011	1.75	1.750	1.750	0.007	0.007
						k = 8									
T	par.	value	mean	median	FSSE	RMSE	value	mean	median	FSSE	RMSE	value	mean	median	FSSE	RMSE
	λ	0.0078	0.022	0.011	0.033	0.036	0.0078	0.013	0.010	0.014	0.015	0.0078	0.011	0.008	0.010	0.010
5000	b	2	2.034	1.869	0.865	0.865	2	1.988	1.952	0.375	0.375	2	2.035	1.998	0.326	0.327
	m_0	1.25	1.247	1.247	0.028	0.028	1.5	1.493	1.493	0.031	0.032	1.75	1.749	1.750	0.025	0.025
	λ	0.0078	0.010	0.008	0.005	0.006	0.0078	0.009	0.008	0.003	0.003	0.0078	0.008	0.008	0.003	0.003
10000	b	2	2.001	1.991	0.262	0.262	2	1.998	1.995	0.150	0.150	2	2.007	2.002	0.109	0.110
	m_0	1.25	1.249	1.249	0.013	0.013	1.5	1.498	1.499	0.013	0.013	1.75	1.749	1.750	0.008	0.008

Note: This table shows results of Monte Carlo experiments for ML estimtation of the parameters of Poisson multifractal processes with $k = 2, 4$, and 8. Sample lengths are $T_1 = 5,000$ and $T_2 = 10,000$. The parameter b is equal to 2, γ is given by $2^{-(k-1)}$,and m_0 assumes values 1.25, 1.5, and 1.75. Displayed are the means, medians and finite sample standard error (FFSE) and root mean squared error (RMSE) of the estimates across 1000 Monte Carlo runs for each setting.