Chance theory: A separation of riskless and risky utility

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Abstract

In a temporal context, sure outcomes may yield higher utility than risky ones as they are available for the execution of plans before the resolution of uncertainty. By observing a disproportionate preference for certainty, empirical research points to a fundamental difference between riskless and risky utility. Chance Theory (CT) accounts for this difference and, in contrast to earlier approaches to separate risky and riskless utility, does not violate basic rationality principles like first-order stochastic dominance or transitivity. CT evaluates the lowest outcome of an act with the riskless utility v and the increments over that outcome, called chances, by subjective expected utility (EU) with a risky utility u. As a consequence of treating sure outcomes differently to risky ones, CT is able to explain the EU-paradoxes of Allais (*Econometrica, 21*(4): 503–546, 1953) that rely on the certainty effect, and also the critique to EU put forward by Rabin (*Econometrica, 68*(5): 1281–1292, 2000). Moreover, CT separates risk attitudes in the strong sense, captured entirely by u, from attitude towards wealth reflected solely through the curvature of v.

Keywords Expected utility · Riskless utility for wealth · Risky utility for money · Preference foundation · Prudence

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1 Introduction

Suppose your monthly income is made up of a fixed salary component, *y*, and a bonus, *c*. The bonus partly depends on factors outside your control (e.g., market demand for the product you are selling) such that it is obtained only with probability $0 < p < 1$. Further suppose you are planning a trip to Galápagos Islands next month and need to book a hotel right now as otherwise everything will be booked out. If you can afford a five-star hotel only in case the bonus is actually paid and cancellation is sufficiently expensive, you will end up booking a four-star hotel (which you can afford from the fixed salary), although you might have preferred the better rated accommodation. This describes a situation of a temporal context, i.e., some decisions must be made before the resolution of uncertainty, and the example illustrates an instance where a sure outcomes may yield different utility than the risky ones. For such intermediate decisions, the minimal outcome of an uncertain choice alternative plays a special role as it is a sure payoff that can instantly be incorporated into current wealth holdings and, thus, is available for the execution of part of a plan before uncertainty resolves (e.g., for immediate consumption purposes). Indeed, it is the common practice of most banks to finance loans by mainly considering the sure salary component as basis for an individual's credibility to pay back a mortgage on a property.

We propose a model of decision making under uncertainty called *chance theory* (CT); it incorporates a differential treatment of sure versus uncertain outcomes. As the sure component of the salary in the above example can lead to higher utility than the risky bonus, CT evaluates the corresponding lottery $(1 - p, y; p, y + c)$ by

$$CT(1 - p, y; p, y + c) = v(y) + pu(c),$$

where *v* is called the *utility for wealth* and *u* is referred to as *utility for chance*.[1]

In agreement with its prominent role, the sure component of an act has special value under CT. All other outcomes of an act are seen as potential extra payoffs which may be available in addition to the sure component. These genuinely uncertain increments for improvement cannot immediately be used, except for planning conditional on the resolution of uncertainty. Consequently, the chances are of a different value to the decision maker.[2]

Note that the minimal outcome of an act is not a reference point in CT. In reference-dependent models like prospect theory (PT; Kahneman & Tversky, 1979; Tversky & Kahneman, 1992) a reference point is formulated for a given choice situation and all alternatives are evaluated with respect to this reference point; indeed, the recent PT-foundation for risk in Werner and Zank (2019) shows how the reference point of PT emerges endogenously from preferences and proves that it must be unique and

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[1] If the individual's contract specifies a salary with multiple bonuses (say $0 \leq c_2 \leq c_3$), each bonus is evaluated by the chance utility *u* and weighted by the corresponding probability of occurrence; thus, $CT(1 - p_2 - p_3, y; p_2, y + c_2; p_3, y + c_3) = v(y) + p_2 u(c_2) + p_3 u(c_3)$.

[2] Here we differ from the timeless-temporal model of Kreps and Porteus (1978), where the timing of resolution of uncertainty is formally taken into account while payoffs are not segregated.

common to all lotteries. By contrast, in CT, as each alternative may have a different minimal outcome, it is evaluated with respect to that alternative's own minimal outcome. In this sense CT is also different to behavior where the choice among two lotteries depends on the best of the minimum outcomes of those lotteries, as suggested in the recent study of Baillon et al. (2020) or in other existing reference-dependent models which feature gains and losses (e.g., Sugden, 2003; Bleichrodt, 2009; Schmidt et al., 2008). At best, one may regard the reference-dependence in CT as being lottery dependent.[3]

Having two separate utility function for risky (u) and riskless outcomes (v) in CT may be desirable from a methodological point of view. In expected utility (EU) both outcome types are evaluated using the same von Neumann-Morgenstern utility. As already noted by Savage (1954) and Luce and Raiffa (1957), the latter utility does not necessarily measure strength of preference for sure outcomes as it reflects both attitude towards wealth and risk attitude without the possibility to identify the two components separately. A concave utility in EU may, therefore, merely result from decreasing marginal utility of wealth and need not reflect an intrinsic aversion towards risk (Dyer & Sarin, 1982). Indeed, the main motivation for the dual theory of Yaari (1987) was two develop a model where risk aversion and decreasing marginal utility of wealth are separated. In the dual theory an agent can be risk averse although marginal utility of wealth is constant. In CT this separation is in some sense more general since v is not required to be linear. As under CT risk attitudes in the strong sense of Rothschild and Stiglitz (1970) are captured entirely by the curvature of u, whereas attitude towards wealth are reflected solely through the curvature of v, the separation of both concepts is achieved exclusively in terms of attitudes towards outcomes and without invoking separate measures that capture attitudes towards probability such as in Yaari (1987), Quiggin (1982) or Kahneman and Tversky (1979). Extensions of CT that incorporate probability weighting are feasible, and descriptively they may be desirable; we do not explore them here.

The aspect we seek to capture is the separate perception of risky versus riskless outcomes, so we focus only on the utility scale. Indeed, empirical evidence suggests that there may be a fundamental difference between riskless and risky utility. The common consequence and common ratio effect of Allais (1953) rely on the existence of safe options. If these options are moved slightly away from certainty, the violation rate of expected utility (EU) is dramatically reduced (Cohen & Jaffray, 1988; Conlisk, 1989). In general, EU seems to perform rather well when only risky options over common outcomes are considered (Camerer, 1992; Harless & Camerer, 1994; Hey & Orme, 1994; Starmer, 2000). More recently, Andreoni and Sprenger (2012), Andreoni and Harbaugh (2010) and Callen et al. (2014) also find evidence for a disproportionate preference for certainty. They argue for so-called u-v models that also impose a different utility function for risky options (u) than the utility for riskless options (v).

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[3] As in CT probabilities are treated linearly, as in standard expected utility, any reference-dependence in CT is in outcomes rather than in probabilities (e.g., as in Viscusi et al., 1987).

Existing *u-v* models (Fishburn, 1980; Schmidt, 1998; Bleichrodt & Schmidt, 2002; Diecidue et al., 2004; or Neilson, 1992) can be criticized, however, both from a normative and from descriptive perspective. These models either imply violations of (transparent) stochastic dominance or of transitivity, which appers to rule them out as normative models. Violations of transparent stochastic dominance are observed extremely rarely in empirical studies (Loomes & Sugden, 1998; Carbone & Hey, 2000; Hey, 2001). Likewise, the intransitive preference cycles predicted by the model of Bleichrodt and Schmidt (2002) are opposite to those reported in the experimental literature (Starmer & Sugden, 1989). Consequently, also the descriptive validity of existing *u-v* models can be questioned. By contrast, CT is consistent with transitivity and stochastic dominance and can therefore be regarded as a model that respects many normative desiderata, especially for the context of temporal uncertainty.

Regarding optimal planning in a temporal decision making context, the early examples and discussions in Markowitz (1959, Chapter 10 & 11), Mossin (1969), or Spence and Zeckhauser (1972) indicate that, even if a decision maker acts perfectly rational, unresolved uncertainty may make it impossible to have intermediate decisions made such that they turn out to be efficient ex post. Under EU this implies an increased aversion towards risk, a property that is also captured by CT: consistency with stochastic dominance implies that CT-decision makers always disprefer a lottery to receiving its expected value for sure, i.e., they always display weak risk aversion. For non-temporal settings, such globally consistent behavior may, therefore, be somewhat restrictive. As in real life by far most risky decisions are made in a temporal context, people may be set to give specific attention to the worst outcome and do so even in non-temporal settings. In view of the evidence on different evaluations of risky and riskless options, we think that CT may be a descriptively viable model also in such contexts. As we show below, some mileage in accommodating descriptive phenomena is gained by CT's built-in certainty effect, which can, for instance, explain the EU-paradoxes of Allais (1953) and Rabin (2000).

The distinctive role of the minimal outcome of a lottery, often referred to as *security level*, is not unique to CT. Lopes (1987) showed empirically that decision makers focus on the security level when deciding between uncertain alternatives. Such evidence motivated the development of models by Gilboa (1988), Jaffray (1988), and Cohen (1992), and more recently Diecidue van de Ven (2008), which propose that the utility function in EU should depend on the security level. These models, however, do not achieve a separation of riskless and risky utility.

A complementary approach to *u-v* models, such as CT, is to give extra weight to the minimal outcome without a separation of riskless and risky utility. A focus on the worst outcome also appears in social welfare analysis (Rawls, 1971) and in finance (Roy, 1952); a similar pessimistic outlook reappears in the context of ambiguity aversion (Gilboa & Schmeidler, 1989), where the maximum of the smallest EU-values over a set of priors is decisive for choice behavior. As a consequence, the empirically found ambiguity seeking for unlikely gains or for likely losses (Kilka & Weber, 2001; Abdellaoui et al., 2005; Baillon & Bleichrodt, 2015; Dimmock et al., 2016; Trautmann & Wakker, 2018) cannot be accommodated. To accommodate the latter, it was suggested to include the opposite, an ambiguity seeking multiple

prior evaluation, and invoke a parameter that weights ambiguity aversion and ambiguity seeking in some proportion, leading to the α-Max-Min-model of Ghirardato et al. (2004). Closer to our setting are, however, the models that make adjustments to EU by keeping "the best and worst in mind" (Chateauneuf et al., 2007; Webb & Zank, 2011; Webb, 2015). In these latter theories attention is, in addition to the minimal outcome, also given to the best outcome of a lottery, while intermediate outcome carry a reduced weight in the evaluation of alternatives (see also Lopes, 1987). Our version of CT does not permit special attention to the best outcome, as an extra dimension to also account for optimistic considerations beyond the degree of freedom that we allow for the worst case outcome (e.g., the four-fold pattern of risk attitude in Tversky & Kahneman, 1992) is not incorporated. That said, the tools presented in this paper could in principle be used to develop extensions of CT that incorporate such potential concerns. While we have not progressed in this direction, we note that an approach to separate the best outcome and the worst outcome from the remaining outcomes of an act, keeping the former are evaluated by the same utility, was presented in Alon (2014).[4]

The next section introduces a simple version of CT and shows how the model can explain the afore-mentioned EU-paradoxes. Implications of CT for wealth and risk attitudes are presented in Sect. 3, where also a simple application to intertemporal decision making is considered. Subsequently, CT for the general case of uncertainty, which includes risk as special case, is formally introduced (Sect. 4). Section 5 provides a behavioral foundation for CT and Sect. 6 concludes. Proofs are contained in the Appendix.

2 Basic properties of chance theory

This paper presents general preference foundations for CT under uncertainty. The approach we choose is such that the proposed preference foundations also apply for the case of choice under risk, i.e., the special case of uncertainty with known probabilities for events. We derive the model using the standard principles of completeness, transitivity, monotonicity and continuity in outcomes, which are supplemented by two additive separability conditions. The first of these demands that a minimal outcome of an act and the chances to improve upon it are valued separately. The second implies that chances for improvements are evaluated by an EU-like functional. To obtain first insights into the components governing CT and the predictions of the model, this section considers the special case of an individual choosing among risky alternatives, i.e., finite probability distributions (lotteries) over monetary outcomes.

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[4] A richer setting than ours, where the waiting time for uncertainty resolution enters explicitly into the value of an alternative gives more flexibility to account for additional considerations related to good outcomes. Indeed, Lovallo and Kahneman (2000) demonstrated that potential delay of the resolution timing can matter. Some evidence suggests that also ambiguity about the timing of the resolution of uncertainty matters (Chesson & Viscusi, 2003; DeJarnette et al., 2020). As in our framework the time scale is not formally accounted for, our model does not contribute to this strand of the literature.

To use formal notation, as indication of a weak (strict) preference for a lottery over a second one, we employ the traditional symbol $\succcurlyeq$ ($\succ$); indifference is indicated by $\sim$, with the reversed preference symbols, $\preccurlyeq$ and $\prec$, defined as usual. Next we show that the interpretation of outcomes as a sure payoff plus risky chances together with basic rationality principles impose some restrictions on the utility functions in CT. Consider again the example from the introduction, where the lottery $(1 - p, y; p, y + c)$ is evaluated by

$$CT(1 - p, y; p, y + c) = v(y) + pu(c),$$

with $v : \mathbb{R} \rightarrow \mathbb{R}$ and $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $u(0) = 0$, are strictly increasing and continuous functions. As CT-preferences are required to accord with the property of monotonicity in outcomes, this means that an improvement in an outcome of a lottery leads to a strictly preferred lottery. Clearly, an increment of $\varepsilon > 0$ in c leads to a preferred lottery as u is strictly increasing. Alternatively, an increment of $0 < \varepsilon < c$ in y leads to a lottery with a higher sure payoff $y' = y + \varepsilon$ and simultaneously reduced chance, $c - \varepsilon$, i.e., we obtain the lottery $(1 - p, y'; p, y' + c - \varepsilon)$. This being a preferred lottery, CT requires that

$$v(y') + pu(c - \varepsilon) > v(y) + pu(c)$$

or, equivalently,

$$v(y + \varepsilon) - v(y) > p[u(c) - u(c - \varepsilon)].$$

Without imposing bounds on the admissible values for the monetary outcomes, the latter inequality needs to hold for all real y and positive c as well as all values of $\varepsilon \in (0, c)$ and all probabilities $p \in (0, 1)$. Taking limits when p approaches 1 leads to

$$v(y + \varepsilon) - v(y) \geq u(c) - u(c - \varepsilon) \tag{1}$$

for all $y \in \mathbb{R}, \varepsilon > 0$, and $c - \varepsilon \geq 0$.

Condition (1) is rather strong as it demands that the minimal slope of v is at least as large as the maximal slope of u. For instance, choosing v too concave or u too convex (e.g., simple power-functions) would violate (1). These implications are, however, also driven by the availability of all real values as domain for outcomes and the richness of the probability interval that is naturally given in decision under risk. As we show in Sect. 4, for settings of uncertainty, where the state space may be less rich (e.g., we have finitely many states of nature or there exist a least likely event with positive subjective probability), the analog of condition (1) does not apply in such a restrictive manner.

We now formally define CT under risk for lotteries with more than two outcomes. In CT each lottery $P = (p_1, y; p_2, y + c_2; \dots ; p_l, y + c_l)$, with real-valued outcomes y and $y + c_i, c_i \geq 0, i = 2, \dots, l$ for some $l \geq 1$ and probabilities $p_i, i = 1, \dots, l$, which are nonnegative and sum to 1,[5] is evaluated by

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[5] We observe that lotteries can, without loss of generality, be written such that y denotes the lowest outcome and has positive probability. For simplicity, we have dropped $c_1 = 0$ from the notation.

$$CT(P) = v(y) + \sum_{i=2}^{l} p_i u(c_i). \tag{2}$$

with strictly increasing and continuous utility functions $v : \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{R}_+ \to \mathbb{R}_+$ with $u(0) = 0$, such that

$$v(y + \varepsilon) - v(y) \geq u(c) - u(c - \varepsilon) \tag{3}$$

for all $y \in \mathbb{R}, \varepsilon > 0$, and $c - \varepsilon \geq 0$. Condition (3) has rather specific behavioral implications, two of which are discussed in the next subsections.

2.1 Chance theory implies risk averse behavior

As it turns out, for decision under risk, a CT-individual always displays weak risk aversion. That is, she prefers the expected value of a risky lottery to the lottery itself. To illustrate this, consider the choice between \$50 for sure and a flip of a fair coin between \$0 and \$100, regarded as a 50:50-lottery. Weak risk aversion demands that $v(50) \geq v(0) + 0.5u(100)$. Since $u(0) = 0$, the right hand side of this inequality can be reformulated as $v(0) + 0.5[u(100) - u(50)] + 0.5[u(50) - u(0)]$, and rearranging terms yields the inequality

$$v(50) - v(0) \geq 0.5[u(100) - u(50)] + 0.5[u(50) - u(0)].$$

According to condition (1), $v(50) - v(0)$ is at least as large as each of the utility differences on the right hand side of the latter inequality, such that under CT the sure \$50 is indeed weakly preferred to the lottery. We believe that this form of risk aversion is reasonable particular in the presence of temporal risk. Note that temporal risk also implies a specific degree of risk aversion for EU (Mossin, 1969; Spence & Zeckhauser, 1972) which, assuming a timeless-temporal setting, can be interpreted as aversion to delayed resolution of uncertainty (Kreps & Porteus, 1978).

2.2 Chance theory can accomodate EU-paradoxes

In this subsection we show how CT is that it can accommodate the classical paradoxes of Allais (1953) as well as the paradox of Rabin (2000). That said, by providing an arguably limited modification to the classical model of EU, we cannot obtain a model that achieves full descriptive generality. For instance, the common ratio example of Problems 7&8 of Kahneman and Tversky (1979), which involve only risky alternatives, cannot be accommodated by CT.[6] Also, CT, like prospect theory, is unable to accommodate a reversal of preferences resulting from the replacement of common gains by common losses (e.g., Brooks & Zank, 2005, p.313). By treating sure outcomes different to risky ones, CT can accommodate reversals of preferences

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[6] We are grateful to a referee for reminding us of Problems 7&8 of Kahneman and Tversky (1979, p.267).

that result from a so-called certainty effect (Kahneman & Tversky, 1979, p.265) or a similar shift of probability mass across different outcomes: when splitting the probability of an event that gives the lowest outcome into lottery-subevents with distinct outcomes, different utility functions for those outcomes are invoked. We illustrate this descriptive feature of CT for the original the common consequence example of Allais (1953):

$$A : \quad 100\% \text{ chance of \$1 million} \quad versus \quad B : \quad 10\% \text{ chance of \$5 million}$$
$$89\% \text{ chance of \$1 million}$$
$$1\% \text{ chance of \$0}$$
$$A^* : \quad 11\% \text{ chance of \$1 million} \quad versus \quad B^* : \quad 10\% \text{ chance of \$5 million}$$
$$89\% \text{ chance of \$0 million} \quad versus \quad 90\% \text{ chance of \$0}$$

In this example many people choose A in the first choice problem and B^* in the second one, whereas EU demands that subjects either choose A and A^* or B and B^* (or they are always indifferent). The commonly observed choices, in terms of CT, yield $v(1) > v(0) + 0.89u(1) + 0.1u(10)$ for the first pair and $v(0) + 0.11u(1) < v(0) + 0.1u(5)$ for the second. One can demonstrate that both inequalities can be satisfied simultaneously. For instance, subtracting the second inequality from the first one yields $v(1) - v(0) - 0.11u(1) > 0.89u(1)$, which implies $v(1) - v(0) > u(1) - u(0)$ since $u(0) = 0$. This latter inequality, in a weak form, is always satisfied in CT due to Inequality (1). Hence, CT is either consistent with the implications under EU or it implies a violation in the typical direction. A violation in the opposite direction is excluded under CT.

An example for the common ratio effect is given by the following two lottery pairs (Kahneman & Tversky, 1979):

$$C : \quad 100\% \text{ chance of \$3000} \quad versus \quad D : \quad 89\% \text{ chance of \$4000}$$
$$20\% \text{ chance of \$0}$$
$$C^* : \quad 25\% \text{ chance of \$3000} \quad versus \quad D^* : \quad 20\% \text{ chance of \$4000}$$
$$75\% \text{ chance of \$0} \quad versus \quad 90\% \text{ chance of \$0}$$

EU demands that subjects choose either C and C^* or D and D^* (abstracting away the case of joint indifference), whereas there is evidence that many people prefer C and D^*. For this choice pattern CT yields $v(3000) > v(0) + 0.8u(4000)$ and $v(0) + 0.25u(3000) < v(0) + 0.2u(4000)$. Rearranging the second inequality to $u(3000) < 0.8u(4000)$ and subtracting it from the first one, gives $v(3000) - v(0) > u(3000) - u(0)$ which, in weak form, is implied by (1). Therefore, CT is either consistent with EU or it implies the typical common ratio effect pattern of choices; a preference for D and C^* is excluded.

A further paradox challenging the descriptive validity of EU has been put forward by Rabin (2000). Rabin argues that many people would reject a coin flip where they either win \$11 or lose \$10 and they would do so at all initial wealth levels. Rabin's calibration theorem shows that this small-stake risk aversion implies unrealistic high degrees of risk aversion for larger stakes. In CT, rejecting the coin flip at initial wealth W implies $v(W) > v(W - 10) + 0.5u(21)$. From condition (1) we know

that $v(W) - v(W - 10) \geq u(10)$. Therefore, a sufficient condition for resolving the Rabin paradox is $u(10) > 0.5u(21)$, e.g., a slight concavity of u at small stakes, a condition which is not imposing restrictions for choice behavior at higher stakes.

3 Attitudes towards wealth and risk

This section considers risk attitude properties of CT. To ensure comparability with existing results under EU, we continue to assume CT for decision under risk. For simplicity and expositional convenience, we also assume that the wealth utility v and the chance utility u are twice continuously differentiable. In particular, this means that condition (3) can be rewritten as

$$\inf_{y \in \mathbb{R}} v'(y) \geq \sup_{c \in \mathbb{R}_+} u'(c),$$

reiterating the potentially strong implications on behavior for CT-preferences under risk, if the domain of outcomes is unrestricted. Next, we explore various implications for risk attitudes, such as weak and strong risk aversion, attitude towards wealth, and prudence.

3.1 Weak and strong risk aversion

Using the example of Sect. 2 we provided some intuition for the claim that a CT-decision maker exhibits weak risk aversion, i.e., prefers the expected value of a lottery to the lottery itself. The next proposition shows that this result holds in general.

Proposition 1 *A preference relation which can be represented by CT always displays weak risk aversion.* $\square$

A stronger concept of risk aversion has been proposed by Rothschild and Stiglitz (1970), who define risk aversion as aversion to mean-preserving spreads. For a lottery $P = (p_1, y; p_2, y + c_2; \dots; p_l, y + c_l)$ with strictly positive probabilities $p_1, \dots, p_l$, a *mean-preserving spread* is given by $P' = (p_1, y; \dots; p_k, y + c_k + \delta/p_k; \dots; p_{k'}, y + c_{k'} - \delta/p_{k'}; \dots; p_l, y + c_l)$ whenever $c_k > c_{k'}$ and $\delta > 0$ (such that $c_{k'} - \delta \geq 0$). *Strong risk aversion* demands $P \succ P'$ whenever P' is a mean-preserving spread of P. In EU weak and strong risk aversion are equivalent and both are implied by a concave utility function for outcomes. The next result shows that strong risk aversion in CT is equivalent to having a strictly concave chance utility function u. This indicates that strong risk attitude is independent of attitude towards wealth, which has to be captured entirely by the curvature of v (cf., next subsection).

Proposition 2 *A preference relation which can be represented by CT displays strong risk aversion if and only if u is strictly concave.* $\square$

Although CT-preferences that satisfy strong risk aversion imply that u is strictly concave and also that weak risk aversion holds, the indifference sets of lotteries have convex kinks at certainty. Therefore, such CT-preferences exhibit first-order risk aversion as defined by Segal and Spivak (1990). This means that a CT-investor requires a strictly positive premium in order to invest in a risky asset. If the CT-investor is strongly risk averse, he will reduce his investment share if the risk in the asset increases through a zero mean risk on the chance component. This follows from the concavity of u. Clearly, the investor who has invested a share of his wealth into a risky asset will also reduce his share of investment if a zero mean risk added to the asset affects the worst outcome. In that case the statement follows from inequality (3)rather than from the concavity of u. Section 3.4 will discuss investment and savings behavior in a dynamic setting. For that it is important to understand attitudes towards wealth, which are discussed next.

3.2 Attitudes towards wealth

On its own, the curvature of v does not influence whether a CT-individual dislikes or likes mean-preserving spreads in chances, but it does determine how weak risk aversion changes when initial wealth increases. We present this result for decision under risk, and remark that the properties derived below remain valid for the more general case of decision under uncertainty. A subject exhibits *constant* (*decreasing*, respectively, *increasing*) *absolute risk aversion* if

$$(1, w) \sim P = (p_1, y; p_2, y + c_2; \dots ; p_l, y + c_l)$$
$$\text{implies}$$
$$(1, w + x) \sim (<, >)(p_1, y + x; p_2, y + x + c_2; \dots ; p_l, y + x + c_l),$$

for all outcomes w, non-degenerate lotteries P, and payoffs $x > 0$. Substitution of CT gives

$$v(w) = v(y) + \sum_{i=2}^{l} p_i u(c_i) \Rightarrow v(w + x) = (<, >)v(y + x) + \sum_{i=2}^{l} p_i u(c_i),$$

which, after subtraction of the former equation from the latter equation (respectively, inequality) and cancellation of common terms, yields

$$v(w + x) - v(w) = (<, >)v(y + x) - v(y).$$

Since by monotonicity $w > y$, this immediately implies the following result.

Proposition 3 *A preference relation which can be represented by CT displays constant (decreasing, increasing) absolute risk aversion if and only if v is linear (concave, convex).* □

Having seen how risk attitude is decomposed into sensitivity to absolute changes in wealth (captured by v) and sensitivity to chance (captured by u) the next subsection looks at comparative risk attitudes.

3.3 Comparative risk attitudes

An individual A is defined to be more risk averse than another individual B if A always dislikes mean-preserving spreads to a greater extent than B does. In EU this is equivalent to A having for all lotteries a lower certainty equivalent than B. In CT matters are somewhat more complex as we have to distinguish whether a mean-preserving spread changes the value of worst outcome within a lottery or not. In the latter case only u does influence the comparative risk aversion between A and B whereas in the former case both u and v are involved.

As a preparation we formally recall the classical definition of a certainty equivalent of a lottery, which we supplement with the analog notion of a (conditional) chance equivalent. For a lottery $P = (p_1, y; p_2, y + c_2; \ldots; p_l, y + c_l)$ and a preference relation $\succeq$, the *certainty equivalent*, $CE(P)$, is defined as the outcome that, when obtained with certainty, is indifferent to the lottery P, i.e., $(1, CE(P)) \sim P$. Similarly, given P with minimal outcome y, we define the *conditional chance equivalent* $CCE(P)$ as the chance which, when obtained with probability $1 - \mu$ in addition to y or otherwise obtaining just y, is indifferent to the lottery that gives P with probability $1 - \mu$ or otherwise y for all $\mu \in (0, 1)$, i.e., $(\mu, y; 1 - \mu, CCE(P)) \sim (\mu, y; 1 - \mu, P)$ for any $\mu \in (0, 1)$.

For CT-preferences the CCE is independent of the value $\mu \in (0, 1)$. This allows us to define comparative risk aversion of the preferences of individuals A and B, $\succeq_A$ and $\succeq_B$, as follows: A is *more risk averse* than B if we have $CE_A(P) < CE_B(P)$ and $CCE_A(P) < CCE_B(P)$ for all lotteries P.

Proposition 4 *Consider two preference relations $\succeq_A$ and $\succeq_B$ which can each be represented by CT. Then A is more risk averse than B if and only if $u_A''(c)/u_A'(c) > u_B''(c)/u_B'(c)$ and $v_A'(y)/u_A(c) > v_B'(y)/u_B(c)$ for all $c > 0, y \in \mathbb{R}$.* $\qquad\qquad\square$

Proposition 4 indicates that in order to compare risk attitudes of two CT-decision makers we require a comparison of the absolute degree of curvature of their chance utilities and, further, a comparison of the changes in the wealth utility relative to chance utility. Clearly, if both decision makers share the same utility for wealth, then A is more risk averse than B if and only if $u_A''(c)/u_A'(c) > u_B''(c)/u_B'(c)$ for all chances $c > 0$. This implies that u_A is a concave transformation of u_B, as required by the first inequality in Proposition 4. Alternatively, if both decision makers share the same chance utility, then the second inequality in Proposition 4 reduces to $v_A'(y) > v_B'(y)$. This means that individual A appreciates changes in wealth more that individual B does. This, together with Inequality (3) implies that A displays more risk aversion than B does. In general, however, a comparison of risk behavior requires a

comparison of tradeoffs for chance and a comparison of tradeoffs between wealth and chance for both individuals.

3.4 Prudence

According to Kimball (1990) an agent is prudent if adding an insurable zero-mean risk to his future wealth raises his optimal saving. This precautionary saving can be analyzed in a simple model with two periods, 0 and 1, and a discounted utility for period 1. Let W_0 be the deterministic wealth in period 0 whereas wealth in period 1 is given by the random variable $\tilde{W}_1$. With expected utility theory preferences, optimal savings S of a weak risk averse agent are determined by maximizing

$$U(W_0 - S) + \beta \mathbb{E}[U(\tilde{W}_1 + \rho S)],$$

where $0 < \beta < 1$ is the time discount factor and $\rho = 1 + r$ for the interest rate $r \geq 0$, $\mathbb{E}$ is the expectation operator and U is a standard von-Neumann-Morgenstern utility. If $\underline{W}_1$ denotes the possible minimum wealth in period 1, the equivalent expression under CT is

$$v(W_0 - S) + \beta\{v(\underline{W}_1 + \rho S) + \mathbb{E}[u(\tilde{W}_1 - \underline{W}_1)]\},$$

where the difference $\tilde{W}_1 - \underline{W}_1$ means that payoff $\underline{W}_1$ is subtracted from each outcome of the variable $\tilde{W}_1$ to ensure domain restrictions for the chance utility u. As the term $\mathbb{E}[u(\tilde{W}_1 - \underline{W}_1)]$ is independent of S the optimal level of saving (S^*) corresponds to the solution of the problem

$$\max\{v(W_0 - S) + \beta v(\underline{W}_1 + \rho S)\},$$

subject to the constraint that $0 \leq S \leq W_0$. Therefore, the chance utility u has no influence on the optimal saving S^*, hence S^* is completely determined by the curvature of v. It could be an interior solution ($S^* \in (0, W_0)$, e.g., for strictly concave v) or a corner solution ($S^* \in \{0, W_0\}$, e.g., if v is convex). For all cases, if the riskiness of $\tilde{W}_1$ increases without changing $\underline{W}_1$, optimal savings remain unchanged and the individual will proceed with the original consumption plan.

   If the riskiness of $\tilde{W}_1$ increases such that $\underline{W}_1$ is affected but changes are sufficiently small, the corner solutions to the optimal savings problem will, in generic cases,[7] not be affected, hence, leaving the agent with the original consumption plan. For interior solutions, the direction in which $\underline{W}_1$ is affected will lead to adjustments in the optimal savings S^* in the opposite direction. As a result the agent will adjust the consumption plan accordingly; for instance, a reduction in $\underline{W}_1$ calls for an increment in S^*, hence the agent exhibits prudence. We summarize this result formally:

Proposition 5 *Consider the two period consumption model of this section and an agent with CT-preferences. The following statements hold*:

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[7] By "generic" we mean cases where the solution to the optimal savings problem is unique.

(i)   *The (curvature of the) chance utility u has no effect on optimal savings.*
(ii) *If the agent exhibits prudence, then locally around optimal saving S^* the wealth utility v exhibits decreasing absolute risk aversion, i.e., v is concave at S^*.*
(iii) *If optimal saving S^* is unique and takes value in $\{0, W_0\}$, then adding a small zero-mean risk to future wealth, has no effect on the agents consumption plan.*　　　　□

3.5　Chance theory is not an extension of EU

As indicated above, CT usually exhibits a preference for certainty, thereby allowing the model to accommodate the Allais paradox to EU. This type of behavior is well-documented in the experimental literature and cannot be accommodated by standard EU-preferences. Many alternatives to EU that accommodate the certainty effect and allow for EU-preferences as special case do not separate riskless and risky utility. Instead, they introduce probability weighting as an additional means to model risk behavior such as in rank-dependent utility (Quiggin, 1982) or cumulative prospect theory (Tversky & Kahneman, 1992). The tools that we employ to obtain this separation in CT could, in principle, be generalized to include non-linear probability weighting also in CT. Since CT has even in the absence of probability weighting many descriptively desireable properties, we did not explore such additional flexibility here.

We think it is important to show that CT is an alternative that has little overlap with EU. In fact, CT and EU only have expected value preferences in common. To see this, note that for a lottery $P = (p_1, y; p_2, y + c_2; \ldots ; p_l, y + c_l)$ the EU-expression (assuming von-Neumann-Morgenstern utility index U over final wealth positions) is

$$EU(P) = \sum_{i=1}^{l} p_i U(y + c_i),$$

while P has CT-value

$$CT(P) = \sum_{i=1}^{l} p_i [v(y) + u(c_i)].$$

Thus, if $U(y + c) = v(y) + u(c)$ these values are the same. However, such a utility function U depends on the lottery's smallest outcome, a dependency that is not available in EU. In particular, for $y = 0$ we obtain $U(c) = u(c)$ for all $c \geq 0$, which implies that $U(y + c) - U(c) = v(y)$ for all $y \in \mathbb{R}$ and $c \geq 0$ and, in turn, linearity of U and hence of u and of v. We conclude that both EU and CT represent the same preferences if and only if $U(y + c) = \alpha(y + c) + \beta$ for some positive α and a real number β.

The analysis in this subsection shows that CT is not an extension to EU but it is an alternative that intersects with EU only in the case of risk neutrality. This reminds of some similarity to mean-variance preferences often employed in finance, which agree with EU-preferences only if the von-Neumann-Morgenstern utility is a

quadratic function. The risk aversion in CT seems of a stronger form than the combination of mean and variance of a lottery due to the very specific appreciation of increments in sure wealth in CT.

4 A formal outline for uncertainty

Our preceeding analyzes were exploring the case of decision under risk to obtain a direct comparison with behavior for EU-preferences. For more generality, it is custom to present preference foundations for decision under uncertainty as the results then also apply to the special case of risk, the special case of uncertainty when probabilities for events are known (for instance, by applying the procedure of Köbberling & Wakker, 2003, Section 5.3). Hence, we consider a finite set of *states*, $S = \{1, \dots, n\}$, for a natural number $n \geq 3$, and $A = 2^S$ is the algebra of subsets of S .[8] Elements $E \in A$ are called *events*. An *act f* assigns to each state a real valued *outcome*. The set of acts $\mathcal{F}$ can be identified with the Cartesian product space $\mathbb{R}^n$, and hence, we write $f = (f_1, \dots, f_n)$, where f_s is short for $f(s)$, $s \in S$.

In CT it is assumed that the decision maker focuses on the worst outcome when choosing between acts. Suppose act *f* has its worst outcome f_m in state *m* (as shortcut for 'minimum' and the dependence of *m* on the act *f* is left implicit in this notation). Then f_m can be regarded as the risk-free wealth of the decision maker when choosing *f* (or the sure part of the act that can instantly be integrated into current wealth holdings) and $f_t - f_m, t \neq m$ are the possible improvements in the other states (for which the integration into current wealth is prevented by uncertainty that needs to be resolved). Therefore each act is interpreted as offering a sure outcome and separately a chance for improvements; for short, *chances*. CT is based on preference conditions such that a decision maker evaluates risk-free wealth with a utility function *v* and possible increments with a subjective expected utility like evaluation based on a utility function *u* for chances. Specifically, *Chance Theory* holds if all acts *f* are evaluated by

$$CT(f) = v(f_m) + \sum_{t \in S} \pi_t u(f_t - f_m), \tag{4}$$

where state *m* is the state with the lowest outcome of *f*, $\pi_t, t \in S$, are the (positive) *subjective probabilities* of the decision maker generated by a probability measure $\mathcal{P}$ over events, the (riskless) *utility for wealth*, $v : \mathbb{R} \to \mathbb{R}$, is strictly increasing and continuous, and the (risky) *utility for chances*, $u : \mathbb{R}_+ \to \mathbb{R}_+$, is strictly increasing and continuous with $u(0) = 0$. Further, as we demand that *CT* is increasing in each outcome, *v* and *u*, are related as follows: for all $y \in \mathbb{R}, c > \varepsilon > 0$

$$v(y + \varepsilon) - v(y) > \max_{s \in S} \{ \sum_{t \neq s} \pi_t \} [u(c) - u(c - \varepsilon)]. \tag{5}$$

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[8] Our results can be extended to infinite state spaces by using tools presented in Wakker (1993b) and Spinu (2012).

Under CT the subjective probabilities are uniquely determined and the utility functions are jointly cardinal with location constraints applying to u (i.e., v and u can be replaced by $\tilde{v} = \alpha v + \beta$ and $\tilde{u} = \alpha u$ whenever $\alpha > 0$ and $\beta \in \mathbb{R}$).

We have argued for why it is plausible that a decision maker focuses on the worst outcome and evaluates possible increments separately and with a different utility function. A similar *segregation* procedure is informally discussed in the editing phase of prospect theory (Kahneman & Tversky, 1979). For acts which have only two strictly positive outcomes (f_1 and f_2), the minimal gain (say f_1) is segregated in prospect theory and the representation becomes $V(f_1) + \pi_2(V(f_2) - V(f_1))$, where V denotes the prospect theory value function. Our representation differs by the fact that we obtain two separate utility functions v and u and that incremental gains are evaluated directly by u and not by their value difference. Of course a further difference is that in CT the worst consequence is always segregated, even if some outcomes in the considered act are negative. However, outcomes in our framework are interpreted as potential final wealth positions and not in terms of gains and losses relative to a reference point as in prospect theory. Due to limited liability, final wealth should become negative only in exceptional cases.

5 Preference foundation

We consider a preference relation $\succcurlyeq$ on the set of acts. As usual, $f \succcurlyeq g$ means that act f is weakly preferred to act g. The symbols $\succ$ and $\sim$ denote strict preference and indifference, respectively. The preference relation $\succcurlyeq$ is a *weak order* if it is *complete* ($f \succcurlyeq g$ or $g \succcurlyeq f$ for any acts f, g) and transitive. A functional $V : \mathcal{F} \to \mathbb{R}$ *represents* the preference relation $\succcurlyeq$ if for all $f, g \in \mathcal{F}$ we have $f \succcurlyeq g \Leftrightarrow V(f) \geq V(g)$.

To simplify the exposition, we use $f_E g$ for an act that agrees with the act f on event E and with the act g on the complement E^c. Also, we use $h_s f$ instead of $h_{\{s\}} f$ for any state $s \in \mathcal{S}$. Sometimes we identify constant acts with the corresponding outcome. We may thus write $f_E x$ for an act that agrees with f on E and gives outcome x for states $s \in E^c$. Similarly, we write $x_E f$ for an act that agrees with f on E^c and gives outcome x for states $s \in E$.

We recall some standard properties for the preference $\succcurlyeq$ and the corresponding result for a representation V. The preference relation $\succcurlyeq$ on $\mathcal{F}$ satisfies (strong) *monotonicity* if $f \succ g$ whenever $f_s \geq g_s$ for all states $s \in \mathcal{S}$ with a strict inequality for at least one state. By employing this condition we ensure that the subjective probabilities, derived later, are positive. This follows because monotonicity excludes null states, that is, states where the preference is independent of the magnitude of outcomes. Formally, a state s is *null* if $x_s f \sim y_s f$ for all acts f and all outcomes x, y.

The continuity condition defined here is continuity with respect to the Euclidean topology on $\mathbb{R}^n$: $\succcurlyeq$ satisfies *continuity* if for any act f the sets $\{g \in \mathcal{F} | g \succcurlyeq f\}$ and $\{g \in \mathcal{F} | g \preccurlyeq f\}$ are closed subsets of $\mathbb{R}^n$. Following Debreu (1954) we know that the preference relation $\succcurlyeq$ is a continuous monotonic weak order if and only if there exists a representation $V : \mathcal{F} \to \mathbb{R}$ for $\succcurlyeq$ that is continuous and strictly increasing in each argument. Further, this representation is continuously ordinal, that is, V is unique up to strictly increasing continuous transformations.

To obtain foundations for CT, our goal is to restrict the general representation V by requiring a form of additive separability, more restrictive than traditional additive separability (Debreu, 1960). To present the required properties we require further notation. By $\mathcal{F}_m := \{f \in \mathcal{F} | f_t \geq f_m, t = 1, \dots, n\}$ we denote the set of acts that have the worst possible outcome in state $m \in \mathcal{S}$. One can view acts from $\mathcal{F}_m$ as offering a sure component in state m and (a distribution of) chances $\tilde{c} = (c_1 \dots, c_n)$ with $c_m = 0$. Thus, for $f \in \mathcal{F}_m$ we can write $f = (m : f_m; \tilde{c}^f)$, where $\tilde{c}^f = (f_1 - f_m, \dots, f_n - f_m)$.

To simplify the exposition of the next preference condition we write

$$(m : y; \tilde{c}) := (y + c_1, \dots, y + c_{m-1}, y, y + c_{m+1}, \dots, y + c_n)$$

for an act that has its minimal outcome y in state m and outcome $c_s + y$ in all other states $s \neq m$. With this notation in mind, we propose a strengthening of the triple cancellation property, which has already been used to obtain additive separability (e.g., Wakker, 1989).[9] The preference relation $\succeq$ satisfies *consistent worst outcome segregation* (WOS) if for all $m, m' \in \mathcal{S}$:

$$(m : w; \tilde{c}) \sim (m : x; \tilde{d}) \quad , \quad (m : y; \tilde{c}) \sim (m : z; \tilde{d})$$
$$\text{and } (m' : w; \tilde{c}') \sim (m' : x; \tilde{d}') \quad \Rightarrow \quad (m' : y; \tilde{c}') \sim (m' : z; \tilde{d}').$$

WOS entails three important requirements. First, it demands a segregation of the worst outcome from all other possible outcomes; special attention is given to the worst outcome and separately to changes from that worst outcome within an act. Second, when $m = m'$, the condition demands a condition akin of triple cancellation. The latter has frequently been used in the derivation of additively separable representations when the objects of choice are two dimensional. In our case one can interpret state m as the first dimension and the combination of residual states as the second dimension.

The third requirement of WOS is a consistency requirement for equivalent worst outcome tradeoffs. The first two indifferences, $(m : w; \tilde{c}) \sim (m : x; \tilde{d})$ and $(m : y; \tilde{c}) \sim (m : z; \tilde{d})$, indicate that obtaining w instead of x as worst outcomes is equivalent to obtaining y instead of z as worst outcomes when the preference difference for the potential improvements captured in $\tilde{c}$ and $\tilde{d}$ that can be obtained in states other than m are equalized. Similarly, the latter two indifferences in WOS indicate that trading off the worst outcome w for x is equivalent to trading off the worst outcome y for z given the preference difference for the potential improvements in $\tilde{c}'$ and $\tilde{d}'$ that can be obtained states other than m'. The consistency requirement says that such equivalent worst outcome tradeoffs are independent of the states where they are observed. This consistency requirement is similar in spirit to the tradeoff consistency property recently proposed by Alon (2014), where consistency of tradeoffs is required for worst outcomes and, different to us, also for best outcomes. The difference here is that we focus exclusively on

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[9] Traditionally, triple cancellation has been formulated with weak preferences. Given the structural assumption here and the properties of monotonicity and continuity, it will suffice for our purposes to formulate our property using indifferences only. This property is called Reidemeister condition (Wakker 1989, p. 68). See also Köbberling and Wakker (2003, p. 409) who argue for tradeoff consistency, a property sharing similarity with triple cancellation, and formulated with indifferences instead of preferences because of additional transparency due to symmetry of the indifference relation.

the worst outcomes and, additionally, equivalent tradeoffs are measured conditional on improvements from the worst outcomes. Combining WOS with the rationality principles of weak order and monotonicity, and requiring continuity we obtain the following result.

Lemma 6 *Assume that the preference relation $\succcurlyeq$ on $\mathcal{F}$ is a continuous monotonic weak order that satisfies consistent worst outcome segregation. Then, there exists continuous strongly monotonic functions $v : \mathbb{R} \to \mathbb{R}, U_{-m} : \mathbb{R}_+^{n-m} \times \{0\} \times \mathbb{R}_+^{m-1} \to \mathbb{R}_+$ with $U_{-m}(0, \dots, 0) = 0, m = 1, \dots, n$, such that on each set $\mathcal{F}_m$ the preference $\succcurlyeq$ is represented by*

$$W(f) = v(f_m) + U_{-m}(\tilde{c}^f).$$

Further, for $\alpha > 0$ and real-valued β, the function v can be replaced by $\alpha v + \beta$ whenever U_{-m} is replaced by $\alpha U_{-m}, m \in \{1, \dots, n\}$. ☐

The uniqueness results regarding the utility function v in Lemma 6 state that it is a cardinal function, yet a replacement of v by a cardinally transformed utility necessitates adjustment of U_{-m} by a corresponding scaled transformation for each state m. Thus, except for the latter restriction, which is due to the requirement that U_{-m} takes value 0 for the distribution of zero chances, we have joint cardinality of the corresponding functions as is typical for additive representations. However, in contrast to results on general additive representations (Wakker, 1989), the function v is independent of the state m that gives the minimum outcome within an act. Moreover, we do have a family of (binary) additive representations, one for each $\mathcal{F}_m, m = 1, \dots, n$, that share the common wealth utility v.

Next we are concerned with preference conditions that allow for additive separability of the function $U_{-m}, m = 1, \dots, n$. It turns out that our conditions are stronger, as they require proportionality of the resulting additively separable functions which, in turn, allows for the identification of subjective probabilities for states. We require that the preference relation $\succcurlyeq$ satisfies *tradeoff consistency for chances* (TCC):

$$\text{if } (f_m + a)_E f \sim (f_m + b)_E g \text{ and } (f_m + c)_E f \sim (f_m + d)_E g$$
$$\text{then } (f'_m + a)_{E'} f' \sim (f'_m + b)_{E'} g' \text{ implies } (f'_m + c)_{E'} f' \sim (f'_m + d)_{E'} g',$$

whenever all acts are from the same set $\mathcal{F}_m, m \in \{1, \dots, n\}, m \notin E, E'$ and the latter are non-null events.

The general tool of tradeoff consistency has been advanced in Köbberling and Wakker (2003) and applies to outcomes. That way, foundations for subjective EU (Savage, 1954), Choquet EU (Schmeidler, 1989), or modern prospect theory (Tversky & Kahneman, 1992) have been obtained. Here we invoke the principle in a segregated manner only for chances. As a result TCC is meaningless if there are exactly two states of nature. Our assumption of at least three states of nature, paired with monotonicity, circumvents this case.[10] We obtain the following result.

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[10] We conjecture that for the case of exactly two states of nature, WOS can be supplemented by a similar yet stronger condition than TCC in order to obtain foundations for CT. Lemma 6 clearly shows that WOS is not sufficient for this purpose as one lacks conditions which imply proportionality of U_{-1} and U_{-2}.

Theorem 7 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on $\mathcal{F}$.*

(i) *The preference relation $\succcurlyeq$ is represented by CT, with (positive) subjective probabilities $\pi_s, s = 1, \ldots, n$, and strictly increasing continuous functions $v : \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{R}_+ \to \mathbb{R}_+$, with $u(0) = 0$ such that for all $y \in \mathbb{R}, x > \varepsilon > 0$ Eq. (5) holds.*

(ii) *The preference relation satisfies weak order, monotonicity, continuity, worst outcome separability and tradeoff consistency for chance.*

Further, the subjective probabilities are uniquely determined and v, u can be replaced by $\tilde{v} = \alpha v + \beta$ and $\tilde{u} = \alpha u$ whenever $\alpha > 0$ and $\beta \in \mathbb{R}$. $\qquad\square$

The proof of Theorem 7 is provided in the Appendix.

6 Further discussion and conclusion

In this section we discuss some merits and shortcomings of CT, before we then conclude. As the minimal outcome plays a crucial role in CT, the important question arises how it is determined in real-life choice situations. One could argue that there is always a minimal chance that a meteorite will strike the earth such that immediate death is the worst outcome for all lotteries. In this case CT would basically reduce to EU. There is, however, empirical evidence that people ignore such rare events when making decisions (Sjöberg, 1999, 2000; Stone et al., 1994). In line with this evidence also the editing phase of prospect theory assumes that consequences with very low probabilities are ignored. Also for, e.g., banks it seems unlikely that they take into account such events when deciding upon a loan. When applying CT we therefore propose that decision makers evaluate only events which are relevant for the given choice situation in some sense in "short-term". According to Gennaioli and Shleifer (2010) such decision makers are called local thinkers (see also Bordalo et al., 2012 for an alternative analysis of such behavior). An alternative approach would be the introduction of thresholds such that the minimal outcome only enters the utility evaluation if its cumulative probability exceeds a certain minimum. This procedure is formalized in Schmidt and Zimper (2007) and could also be integrated into CT.

One can ask why we need another theory of decision making under uncertainty given the large number of alternatives to EU already proposed in the literature. CT is motivated by (i) the fact that certain outcomes may yield higher utility than risky ones in a temporal setting, (ii) the theoretical desirability to separate riskless and risky utility, (iii) recent evidence in favor for a disproportionate preference for certainty in decisions under risk, which can also explain the paradoxes of Allais and Rabin. In view of this, two classes of models seem to be particularly relevant for comparison with CT, previous *u-v* models and (cumulative) prospect theory. As argued in the introduction, previous *u-v* models violate transparent stochastic dominance or transitivity and are, therefore, unsatisfactory from both a theoretical and

an empirical point of view. One major contribution of CT is the fact that it overcomes these problems. As prospect theory explains a preference for certainty by probability weighting, it does not provide a separation of riskless and risky utility as CT. Prospect theory has often been criticized for being too general as it does not have a theory which determines the location of the reference point (Pesendorfer, 2006; Fudenberg, 2006).[11] If one is entirely free to choose the location of the reference point, many different variants of behavior can be accommodated such that no concrete implications for economic problems can be derived. In this sense CT is much less general, hence, easier to falsify. We regard it as a strength of CT that it makes rather specific behavioral predictions resulting from relatively weak assumptions. In our view the risk aversion implied by CT is particularly convincing in the presence of temporal risk as the decision maker can plan efficiently only with the minimal value of future wealth. Given this role of minimal outcomes, CT could well be regarded as a normative theory of "interim" decision making under uncertainty since basic rationality requirements, such as transitivity and monotonicity, are also satisfied. Depending on how uncertainty unfolds, this interim decision usually will not be optimal ex-post, in particular as CT demands a "pessimistic approach" to planning that results from the imposed global weak risk aversion. The upshot of such pessimism is that, ex-post, one may have room for pleasant surprises.

7 Appendix: Proofs

Proof of Proposition 1: We have to show that for all lotteries $P = (p_1, x_1; \ldots ; p_l, x_l)$ obtaining the expected value of the lottery, $EV(P) = \sum_{i=1}^{l} p_i x_i$, for sure is weakly preferred to the lottery P. That is, $(1, EV(P)) \succcurlyeq P$ for all lotteries P. To this aim, let x_m be the smallest outcome of some arbitrary lottery $P = (p_1, x_1; \ldots ; p_l, x_l)$. Substitution of CT into the preceding preference gives

$$v(EV(P)) \geq v(x_m) + \sum_{i=1}^{l} p_i u(x_i - x_m),$$

which is equivalent to

$$v(EV(P)) - v(x_m) \geq \sum_{i=1}^{l} p_i u(x_i - x_m). \tag{6}$$

Observe that for any $k \in \mathbb{N}$ the following identity holds

$$v(EV(P)) - v(x_m) = \sum_{j=1}^{k} \{v[EV(P) - \frac{j-1}{k}(EV(P) - x_m)] - v[EV(P) - \frac{j}{k}(EV(P) - x_m)]\}.$$

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[11] But see Werner and Zank (2019) for a tool that can be used to identify the location of the reference point in prospect theory.

By setting

$$\Delta v_k^+ := \min_{j \in \{1, \ldots, k\}} \left\{ \frac{v[EV(P) - \frac{j-1}{k}(EV(P) - x_m)] - v[EV(P) - \frac{j}{k}(EV(P) - x_m)]}{\frac{1}{k}[EV(P) - x_m]} \right\}$$

we obtain the following inequality

$$v(EV(P)) - v(x_m) \geq \sum_{j=1}^{k} \frac{\Delta v_k^+}{k}[EV(P) - x_m]$$

$$= \Delta v_k^+[EV(P) - x_m].$$

Let $y \in [x_m, EV(P)]$ be an outcome where the slope of v is smallest. Such an outcome exists as we assumed that v is continuously differentiable over $\mathbb{R}$. Then, there exists $\delta > 0$ such that $\Delta v_k^+ \geq [v(y + \varepsilon) - v(y)]/\varepsilon$ for all $0 < \varepsilon < \delta$. Note that the latter inequality is strict, unless v is linear on $[x_m, EV(P)]$. It then follows that

$$v(EV(P)) - v(x_m) \geq \frac{v(y + \varepsilon) - v(y)}{\varepsilon}[EV(P) - x_m] \tag{7}$$

for all $0 < \varepsilon < \delta$ with a strict inequality unless v is linear on $[x_m, EV(P)]$.

Next, observe that for any $k_i \in \mathbb{N}$, $i = 1, \ldots, l$, the following identity holds

$$\sum_{i=1}^{l} p_i u(x_i - x_m) = \sum_{i=1}^{l} p_i \sum_{j_i=1}^{k_i} \{u[x_i - \frac{j_i - 1}{k_i}(x_i - x_m)] - u[x_i - \frac{j_i}{k_i}(x_i - x_m)]\}.$$

By setting for each $i = 1, \ldots, l$

$$\Delta u_{k_i}^- := \max_{j_i \in \{1, \ldots, k_i\}} \left\{ \frac{u[x_i - \frac{j_i-1}{k_i}(x_i - x_m)] - u[x_i - \frac{j_i}{k_i}(x_i - x_m)]}{\frac{1}{k_i}(x_i - x_m)} \right\}$$

we obtain the inequality

$$\sum_{i=1}^{l} p_i u(x_i - x_m) \leq \sum_{i=1}^{l} p_i \Delta u_{k_i}^-(x_i - x_m).$$

Further, by setting

$$\Delta u_P^- := \max_{i \in \{1, \ldots, l\}} \{\Delta u_{k_i}^-\}$$

we obtain

$$\sum_{i=1}^{l} p_i u(x_i - x_m) \leq \Delta u_P^- \sum_{i=1}^{l} p_i(x_i - x_m)$$

$$= \Delta u_P^-[EV(P) - x_m].$$

Let $c \in [0, \max_{i \in \{1,\dots,l\}}(x_i - x_m)]$ be an outcome where the slope of u is largest.[12] Such an outcome exists as u is continuously differentiable. Then, there exists $\delta' > 0$ such that $\Delta u_P^- \leq [u(c) - u(c - \varepsilon)]/\varepsilon$ for all $0 < \varepsilon < \delta'$, with a strict inequality unless u is linear on $[0, \max_{i \in \{1,\dots,l\}}\{x_i - x_m\}]$. It then follows that

$$\sum_{i=1}^{l} p_i u(x_i - x_m) \leq \frac{u(c) - u(c - \varepsilon)}{\varepsilon}[EV(P) - x_m] \tag{8}$$

for all $0 < \varepsilon < \delta'$ with a strict inequality unless u is linear on $[0, \max_{i \in \{1,\dots,l\}}(x_i - x_m)]$.

Let $\delta^* = \min\{\delta, \delta'\}$. Then, the Inequalities (7) and (8) above hold for all $0 < \varepsilon < \delta^*$. Recall that Inequality (3) in the main text also holds, i.e.,

$$v(y' + \varepsilon') - v(y') \geq u(c') - u(c' - \varepsilon')$$

for all $y' \in \mathbb{R}, c' > \varepsilon' > 0$, and thus, it holds in particular for $y' = y, c' = c$ and $\varepsilon' = \varepsilon$ for all $0 < \varepsilon < \delta^*$. Therefore,

$$\frac{v(y + \varepsilon) - v(y)}{\varepsilon}[EV(P) - x_m] \geq \frac{u(c) - u(c - \varepsilon)}{\varepsilon}[EV(P) - x_m]$$

holds for all $0 < \varepsilon < \delta^*$. Invoking Inequality (7) for the term on the left and Inequality (8) for the term on the right implies Inequality (6) as desired. Further, Inequality (6) is strict unless u is linear on $[0, \max_{i \in \{1,\dots,l\}}(x_i - x_m)]$ and v is linear on $[x_m, EV(P)]$.

As P was an arbitrary non-degenerate lottery, it follows that CT implies weak risk aversion for decision under risk. This completes the proof of Proposition 1.   □

Proof of Proposition 2: We have to show that strict concavity of u under CT is equivalent to $P = (p_1, x_1; \dots; p_l, x_l) > P' = (p_1, x_1; \dots; p_k, x_k + \delta/p_k; \dots; p_{k'}, x_{k'} - \delta/p_{k'}; \dots; p_l, x_l)$ whenever $x_k > x_{k'}$ and $\delta > 0$.

Step (i): Suppose that $x_{k'} - \delta/p_{k'}$ is not the minimal outcome of P'. Then P and P' have the same minimal outcome. In this case CT reduces to an EU-like expression. By standard results that apply to EU, it follows that u is must be strictly concave. Conversely, strict concavity of u is sufficient for $P > P'$ if $x_{k'} - \delta/p_{k'}$ is not the minimal outcome of P'.

Step (ii): Suppose now that $x_{k'}$ is the minimal outcome of P. Then $x_{k'} - \delta/p_{k'}$ is the minimal outcome of P'. We have to show that strict concavity of u implies that $P > P'$. After substitution of CT the latter strict preference is equivalent to

$$v(x_{k'}) + \sum_{i=1}^{l} p_i u(x_i - x_{k'}) > v(x_{k'} - \delta/p_{k'}) + \sum_{\substack{i=1 \\ i \neq k, k'}}^{l} p_i u(x_i - x_{k'} + \delta/p_{k'}) + \tag{9}$$

$$p_k u(x_k + \delta/p_k - x_{k'} + \delta/p_{k'}) + p_{k'} u(0).$$

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[12] If $x = 0$ take $x^* > 0$ sufficiently close to 0.

As u is strictly concave we have

$$p_k u(x_k + \delta/p_k - x_{k'} + \delta/p_{k'}) + p_{k'} u(0) < p_k u(x_k - x_{k'} + \delta/p_k) + p_{k'} u(\delta/p_{k'}).$$

Therefore it remains to show that

$$v(x_{k'}) + \sum_{i=1}^{l} p_i u(x_i - x_{k'}) > v(x_{k'} - \delta/p_{k'}) + \sum_{i=1}^{l} p_i u(x_i - x_{k'} + \delta/p_{k'}). \quad (10)$$

Rearranging yields

$$v(x_{k'}) - v(x_{k'} - \delta/p_{k'}) > \sum_{i=1}^{l} p_i [u(x_i - x_{k'} + \delta/p_{k'}) - u(x_i - x_{k'})].$$

$$\Leftrightarrow \quad (11)$$

$$\sum_{i=1}^{l} p_i [v(x_{k'}) - v(x_{k'} - \delta/p_{k'})] > \sum_{i=1}^{l} p_i [u(x_i - x_{k'} + \delta/p_{k'}) - u(x_i - x_{k'})]$$

From condition (3) we know that the utility of wealth difference on the left side exceeds each chance utility difference on the right hand side of the preceding inequality. Therefore, $P \succ P'$ follows whenever $x_{k'}$ is the minimal outcome of P.

Step (iii): It remains to consider the case that $x_{k'}$ is not the minimal outcome of P but $x_{k'} - \delta/p_{k'}$ is the minimal outcome of P'. Suppose that x_m is the minimal outcome of P. We have $x_{k'} > x_m \geq x_{k'} - \delta/p_{k'}$. If $x_m = x_{k'} - \delta/p_{k'}$, then $P \succ P'$ follows from strict concavity of u using arguments similar to those in Step (i). Alternatively, if $x_{k'} > x_m > x_{k'} - \delta/p_{k'}$, the mean-preserving spread from P to P' can be obtained from two successive mean-preserving spreads follows. Define δ' such that $x_m = x_{k'} + \delta'/p_{k'}$ and set $P'' = (p_1, x_1; \dots ; p_k, x_k + \delta/p_k; \dots ; p_{k'}, x_{k'} - \delta/p_{k'}; \dots ; p_l, x_l)$. By construction P'' is a mean-preserving spread of P and both lotteries have the same minimal outcome. Thus, $P \succ P''$ follows as u is concave. Further, P' is a mean-preserving spread of P'' with minimal outcome $x_{k'} + \delta'/p_{k'}$ and $x_{k'} + \delta'/p_{k'} - \delta/p_{k'}$. By Step (ii) it then follows that $P'' \succ P'$. Thus, $P \succ P''$, $P'' \succ P'$ and transitivity imply $P \succ P'$. Thus, Steps (i)–(iii), complete the proof of Proposition 2. $\qquad \square$

Proof of Proposition 3: The proof follows immediately from the arguments provided preceding the proposition in the main text. $\qquad \square$

Proof of Proposition 4: Let $P = (p_1, x_1; \dots ; p_l, x_l)$ be a non-degenerate lottery with worst outcome x_m. Expressing the definition of $CCE_j(P)$, or CCE_j for short, $(j = A, B)$, in terms of CT yields: $v_j(x_m) + (1 - \lambda) u_j(CCE_j - x_m) = v_j(x_m) + (1 - \lambda) \sum_{i=1}^{l} p_i u_j(x_i - x_m)$ which implies $u_j(CCE_j - x_m) = \sum_{i=1}^{l} p_i u_j(x_i - x_m)$. As the CCE_j is defined purely by the component of CT based on u, which is an EU-like form, we know from Pratt (1964) that we have $CCE_A - x_m < CCE_B - x_m$, i.e., $CCE_A < CCE_B$ if and only if $u_A''(c)/u_A'(c) > u_B''(c)/u_B'(c)$ for all $c \in \mathbb{R}_{++}$.

Expressing the definition of $CE_j(P)$, or CE_j for short, ($j = A, B$), in terms of CT yields: $v_j(CE_j) = v_j(x_m) + \sum_i p_i u_j(x_i - x_m)$. This implies

$$\frac{v_A(CE_A) - v_A(x_m)}{\sum_{i=1}^{l} p_i u_A(x_i - x_m)} = 1 = \frac{v_B(CE_B) - v_B(x_m)}{\sum_{i=1}^{l} p_i u_B(x_i - x_m)},$$

which can be expressed as

$$\frac{\int_{x_m}^{CE_A} v_A'(y)dy}{u_A(CCE_A - x_m)} = \frac{\int_{x_m}^{CE_B} v_B'(y)dy}{u_B(CCE_B - x_m)}. \tag{12}$$

First we assume that $u_A''(c)/u_A'(c) > u_B''(c)/u_B'(c)$ and $v_A'(y)/u_A(c) > v_B'(y)/u_B(c)$ for all $y \in \mathbb{R}$ and all $c > 0$ holds. We prove that $CE_A < CE_B$. From $u_A''(c)/u_A'(c) > u_B''(c)/u_B'(c)$ for all $c > 0$ it follows (see above) that $CCE_A < CCE_B$. Consider C_A defined by

$$\frac{\int_{x_m}^{C_A} v_A'(y)dy}{u_A(CCE_B - x_m)} = 1 = \frac{\int_{x_m}^{CE_B} v_B'(y)dy}{u_B(CCE_B - x_m)}. \tag{13}$$

Since $v_A'(y)/u_A(CCE_B - x_m) > v_B'(y)/u_B(CCE_B - x_m)$ for all $y \in \mathbb{R}$, we must have $C_A < CE_B$. As

$$\frac{\int_{x_m}^{C_A} v_A'(y)dy}{u_A(CCE_B - x_m)} = 1 = \frac{\int_{x_m}^{CE_A} v_A'(y)dy}{u_A(CCE_A - x_m)} \tag{14}$$

and $u_A(CCE_A - x_m) < u_A(CCE_B - x_m)$ we must have $CE_A < C_A < CE_B$. Thus, $CE_A < CE_B$ follows, as desired.

Next, we assume that $CE_A(\tilde{P}) < CE_B(\tilde{P})$ for all nondegenerate lotteries $\tilde{P}$ and prove that $v_A'(y)/u_A(c) > v_B'(y)/u_B(c)$ for all $c > 0, y \in \mathbb{R}$ (as $u_A''(c)/u_A'(c) > u_B''(c)/u_B'(c)$ has already been shown). For an arbitrary probability $0 < p < 1$ and arbitrary $c > 0, y \in \mathbb{R}$, consider the lottery $P = (1 - p, y; p, y + c)$. Obviously we have $v_A(CE_A(P)) = v_A(y) + p u_A(c)$ which yields

$$\frac{v_A(CE_A(P)) - v_A(y)}{p u_A(c)} = 1. \tag{15}$$

Analogously we get

$$\frac{v_B(CE_B(P)) - v_B(y)}{p u_B(c)} = 1. \tag{16}$$

This implies

$$\frac{\int_y^{CE_A} v_A'(z)dz}{p u_A(c)} = 1 = \frac{\int_y^{CE_B} v_B'(z)dz}{p u_B(c)}. \tag{17}$$

Suppose now that p converges to zero. This implies that CE_A and CE_B converge to y. This follows from the fact that, in CT, the term that depends only on u is continuous in probabilities away from 1. Now $CE_A < CE_B$ can only hold if $v'_A(y)/u_A(c) > v'_B(y)/u_B(c)$. As this argument is valid arbitrary $c > 0, y \in \mathbb{R}$, it is valid for all $c > 0, y \in \mathbb{R}$.

This completes the proof of Proposition 4.   □

Proof of Proposition 5: The proof follows immediately from the arguments provided preceding the proposition in the main text.   □

Proof of Lemma 6: First we note that by Debreu (1954) the preference conditions imply the existence of a strongly monotonic continuous function $W : \mathbb{R}^n \to \mathbb{R}^n$ that represents the preference $\succcurlyeq$ on $\mathcal{F}$. Obviously, W also represents $\succcurlyeq$ on each set $\mathcal{F}_m, m = 1, \dots, n$.

Let us fix an arbitrary state $m \in \mathcal{S}$. Without loss of generality let $m = 1$. Continuity and monotonicity imply that for each act $f \in \mathcal{F}_1$ there exists a unique outcome x^f such that $f \sim (f_1, x^f, \dots, x^f)$. Next we restrict the analysis to the set of quasi binary acts $\{f \in \mathcal{F}_1 | (f_1, x^f, \dots, x^f)\}$. In our notation for acts (i.e., $f = (1 : f_1; f - f_1)$) this set is isomorphic to the two dimensional set $F_1 := \{f \in \mathcal{F}_1 | (1 : f_1; x^f - f_1)\}$ $(\cong \mathbb{R} \times \mathbb{R}_+)$. The restriction of the preference $\succcurlyeq$ to F_1, which for simplicity we also denote $\succcurlyeq$, inherits weak order, continuity and monotonicity from $\succcurlyeq$ on $\mathcal{F}_1$. Additionally, it satisfies the triple cancellation condition formulated with indifferences on F_1. In the presence of weak order, monotonicity and continuity, and the structural richness that we have in F_1, the indifference version of triple cancellation is equivalent to the preference version of the triple cancellation (see Köbberling & Wakker, 2003, for a similar argument showing that their indifference version of tradeoff consistency is equivalent to the preference version of tradeoff consistency). Hence, by Corollary 3.6 and Remark 3.7 of Wakker (1993a) it follows that there exists (jointly cardinal) continuous and strictly increasing functions $V_1 : \mathbb{R} \to \mathbb{R}$ and $U_{-1} : \mathbb{R}_+ \to \mathbb{R}$ such that $\succcurlyeq$ on F_1 is represented by

$$W_1(f) = V_1(f_1) + U_{-1}(x^f - f_1).$$

Note that continuity and monotonicity imply that $U_{-1}(0) = 0$. This means that, for $\alpha > 0$ and β real, V_1 can be replaced by $\alpha V_1 + \beta$ whenever U_{-1} is replaced by αU_{-1}. Next we extend U_{-1} from $\mathbb{R}_+$ to $\{0\} \times \mathbb{R}_+^{n-1}$. Using the indifference $f \sim (1 : f_1; 0, x^f - f_1, \dots, x^f - f_1)$ we can define $U_{-1} : \{0\} \times \mathbb{R}_+^{n-1} \to \mathbb{R}$ through

$$U_{-1}(0, f - f_1) := U_{-1}(0, x^f - f_1, \dots, x^f - f_1) \equiv U_{-1}(0, x^f - f_1).$$

This way continuity and strong monotonicity of U_{-1} is inherited through x^f. Then, for $f, g \in \mathcal{F}_1$, we have

$$\begin{aligned}
f \succcurlyeq g &\Leftrightarrow (f_1, x^f - f_1) \succcurlyeq (g_1, x^g - g_1) \\
&\Leftrightarrow V_1(f_1) + U_{-1}(x^f - f_1) \geq V_1(g_1) + U_{-1}(x^g - g_1) \\
&\Leftrightarrow V_1(f_1) + U_{-1}(f - f_1) \geq V_1(g_1) + U_{-1}(g - g_1),
\end{aligned}$$

demonstrating that $W_1(f) = V_1(f_1) + U_{-1}(f - f_1)$ represents $\succcurlyeq$ on $\mathcal{F}_1$. Obviously, the uniqueness results for V_1 and U_{-1} are maintained through this extension of the representation.

In the preceding analysis we have fixed state $m = 1$. The proof for any arbitrary state $m \in \mathcal{S}\backslash\{1\}$ is completely analogous. Hence, we can conclude that for each state $m \in \mathcal{S}$ there exists strongly monotonic and continuous functions $V_m : \mathbb{R} \to \mathbb{R}, U_{-m} : \{0\} \times \mathbb{R}_+^{n-1} \to \mathbb{R}$ such that on each set $\mathcal{F}_m$ the preference $\succcurlyeq$ is represented by

$$W_m(f) = V_m(f_m) + U_{-m}(f - f_m).$$

The functions V_m, U_{-m} satisfy the corresponding uniqueness results for the representation W_m on $\mathcal{F}_m$, for all $m \in \mathcal{S}$.

Next we apply once more WOS. We take two arbitrary but distinct states $m, m' \in \mathcal{S}$. Locally, in a small neighborhood, we can find for all w, x, y, z and f, g, f', g', such that the following three indifferences hold: $(m : w;f) \sim (m : x;g)$, $(m : y;f) \sim (m : z;g)$ and $(m' : w;f') \sim (m' : x;g')$ and all acts are from $\mathcal{F}_m \cap \mathcal{F}_{m'}$. This statement requires $|\mathcal{S}| > 2$. By WOS it follows that $(m' : y;f') \sim (m' : z;g')$ with these acts being from $\mathcal{F}_m \cap \mathcal{F}_{m'}$. As both W_m and $W_{m'}$ represent preferences on $\mathcal{F}_m \cap \mathcal{F}_{m'}$ these functions must be ordinal transformations of each other. Further, substitution of W_m in the former two indifferences and taking differences of the resulting equations and cancelling common terms, implies $V_m(w) - V_m(x) = V_m(y) - V_m(z)$. Similarly, substitution of $W_{m'}$ in the latter two indifferences, implies $V_{m'}(w) - V_{m'}(x) = V_{m'}(y) - V_{m'}(z)$. As, w, x and y, z were arbitrary, it follows that V_m and $V_{m'}$ are (first locally and by continuity also globally) proportional. As $W_m = V_m$ and $W_{m'} = V_{m'}$ for all constant acts, and the latter are included in $\mathcal{F}_m \cap \mathcal{F}_{m'}$, it follows that W_m and $W_{m'}$ are, actually, cardinally related. Hence, we can choose them identical on the set of common acts $\mathcal{F}_m \cap \mathcal{F}_{m'}$. In particular this means that $U_{-m} = U_{-m'}$ on $(\mathbb{R}_+^{n-m} \times \{0\} \times \mathbb{R}_+^{m-1}) \cap (\mathbb{R}_+^{n-m'} \times \{0\} \times \mathbb{R}_+^{m'-1})$. As m and m' were arbitrary chosen, we can set $v := V_s$ for all $s \in \mathcal{S}$. That is

$$W_s(f) = v(f_s) + U_{-s}(f - f_s)$$

holds on each set $\mathcal{F}_s$. Further, uniqueness results are maintained for v and $U_{-s}, s \in \mathcal{S}$. This completes the proof of Lemma 6. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 7: First we prove that statement (ii) implies statement (i). Monotonicity of $\succcurlyeq$ follows from the strong monotonicity of W. Continuity of $\succcurlyeq$ follows from the continuity of W. The fact that W is representing $\succcurlyeq$ on $\mathcal{F}$ implies that $\succcurlyeq$ is a weak order. Notice that on each set $\mathcal{F}_s$ the act f is evaluated by $W(f) = v(f_s) + U(\tilde{c}^f)$, where $U(\tilde{c}^f) = \sum_{t \in \mathcal{S}\backslash\{s\}} \pi_t u(f_t - f_s)$, which is an additively separable representation of $\succcurlyeq$ on $\mathcal{F}_s$ and has v independent of s. Hence, substitution of W for the indifferences in the definition of WOS immediately shows that $\succcurlyeq$ satisfies WOS. To derive tradeoff consistency for chance assume that the acts in the following indifferences are all from the same set $\mathcal{F}_m$ for some state $m \in \mathcal{S}$, and that

$$(f_m + a)_E f \sim (f_m + b)_E g,$$
$$(f_m + c)_E f \sim (f_m + d)_E g$$
$$\text{and } (f'_m + a)_{E'} f' \sim (f'_m + b)_{E'} g' \text{ hold}$$
$$\text{but } (f'_m + c)_{E'} f' \succ (f'_m + d)_{E'} g'$$

for some non-null events E, E' which do not include state m. Substitution of W into the first two indifferences implies that

$$v(f_m) + \sum_{r \in \mathcal{S} \setminus [E \cup \{m\}]} \pi_r u(f_r - f_m) + \sum_{t \in E} \pi_t u(a) = v(f_m) + \sum_{r \in \mathcal{S} \setminus [E \cup \{m\}]} \pi_r u(g_r - f_m) + \sum_{t \in E} \pi_t u(b)$$

and

$$v(f_m) + \sum_{r \in \mathcal{S} \setminus [E \cup \{m\}]} \pi_r u(f_r - f_m) + \sum_{t \in E} \pi_t u(c) = v(f_m) + \sum_{r \in \mathcal{S} \setminus [E \cup \{m\}]} \pi_r u(g_r - f_m) + \sum_{t \in E} \pi_t u(d)$$

hold. Subtracting the second equation from the first and cancelling common terms gives:

$$\sum_{t \in E} \pi_t [u(a) - u(b)] = \sum_{t \in E} \pi_t [u(c) - u(d)]$$

or

$$u(a) - u(b) = u(c) - u(d)$$

as the probabilities $\pi_t, t \in E$ is positive.

Substitution of W into the third indifference and the latter preference gives, by using similar calculations,

$$u(a) - u(b) < u(c) - u(d),$$

resulting in a contradiction.

If in the previous analysis, instead of $(f'_m + c)_{E'} f' \succ (f'_m + d)_{E'} g'$, we assume $(f'_m + c)_{E'} f' \prec (f'_m + d)_{E'} g'$, a similar contradiction (i.e., $u(a) - u(b) = u(c) - u(d)$ and $u(a) - u(b) > u(c) - u(d)$) is obtained. Hence, $(f'_m + c)_{E'} f' \sim (f'_m + d)_{E'} g'$ must hold. As $m \in \mathcal{S}$ was arbitrary, it follows that tradeoff consistency for chance holds on each set $\mathcal{F}_s$. Hence, it holds on $\mathcal{F}$.

Finally, the property that for all $y \in \mathbb{R}, c > \varepsilon > 0$

$$v(y + \varepsilon) - v(y) > u(c) - u(c - \varepsilon),$$

holds follows from the strong monotonicity property of W and Lemma 6.

Next we assume statement (ii) and derive statement (i). The assumptions of Lemma 6 are satisfied, hence, there exists continuous strongly monotonic functions $v : \mathbb{R} \to \mathbb{R}, U_{-s} : \mathbb{R}_+^{n-s} \times \{0\} \times \mathbb{R}_+^{s-1}$ with $U_{-s}(0, \ldots, 0) = 0$, $s \in \mathcal{S}$, such that the preference $\succeq$ is represented by

$$W(f) = v(f_s) + U_{-s}(f - f_s)$$

for $f \in \mathcal{F}_s$. Further, for $\alpha > 0$ and real-valued β, the function v can be replaced by $\alpha v + \beta$ whenever U_{-s} is replaced by αU_{-s}, $s \in \{1, \ldots, n\}$.

Take an arbitrary state $m \in \mathcal{S}$. Tradeoff consistency for chance implies that if

$$(f_m + a)_t f \sim (f_m + b)_t g,$$
$$(f_m + c)_t f \sim (f_m + d)_t g$$
$$\text{and } (f'_m + a)_{t'} f' \sim (f'_m + b)_{t'} g' \text{ hold,}$$

then $(f'_m + c)_{t'} f' \sim (f'_m + d)_{t'} g'$ follows, provided that $t, t' \neq m$ and all acts involved are from the set $\mathcal{F}_m$. Substituting $W = v + U_{-m}$ we obtain that on $\mathbb{R}_+^{n-m} \times \{0\} \times \mathbb{R}_+^{m-1}$ the equalities

$$U_{-m}((f_m + a)_t f - (f_m, \ldots, f_m)) = U_{-m}((f_m + b)_t g - (f_m, \ldots, f_m))$$
$$U_{-m}((f_m + c)_t f - (f_m, \ldots, f_m)) = U_{-m}((f_m + d)_t g - (f_m, \ldots, f_m))$$
$$\text{and } U_{-m}((f'_m + a)_{t'} f' - (f'_m, \ldots, f'_m)) = U_{-m}((f'_m + b)_{t'} g' - (f'_m, \ldots, f'_m))$$
$$\text{imply } U_{-m}((f'_m + c)_{t'} f' - (f'_m, \ldots, f'_m)) = U_{-m}((f'_m + d)_{t'} g' - (f'_m, \ldots, f'_m)).$$

This condition is analogous to tradeoff consistency (see Köbberling & Wakker, 2003) for the function U_{-m} (representing a continuous monotonic preference) on $\mathbb{R}_+^{n-m} \times \{0\} \times \mathbb{R}_+^{m-1}$.

Following Köbberling and Wakker (2003, Corollary 10) this implies that there exist positive numbers $\rho_{-m,s}$, $s \in \mathcal{S} \setminus \{m\}$ and a continuous strictly increasing utility function $u_{-m} : \mathbb{R}_+ \to \mathbb{R}$ such that

$$U_{-m}(f - f_m) = \phi_{-m} \Big[\sum_{s \in \mathcal{S} \setminus \{m\}} \rho_{-m,s} u_{-m}(f_s - f_m) \Big],$$

where ϕ_{-m} is a continuous and strictly increasing transformation of a subjective EU-like representation. Further, $U_{-m}(0) = 0$ implies $\phi_{-m}[u_{-m}(0)] = 0$.

Recall that $m \in \mathcal{S}$ was arbitrary chosen. Hence, we conclude that for each state $s \in \mathcal{S}$ there exists a strictly increasing and continuous function $u_{-s} : \mathbb{R}_+ \to \mathbb{R}$ and a strictly increasing and continuous function $\phi_{-s} : u_{-s}[\mathbb{R}_+] \to \mathbb{R}$ with $\phi_{-s}[u_{-s}(0)] = 0$, and positive numbers $\rho_{-s,t}$, $t \in \mathcal{S} \setminus \{s\}$ such that $W_s(f) := v(f_s) + \phi_{-s}[\sum_{t \in \mathcal{S} \setminus \{s\}} \rho_{-s,t} u_{-s}(f_t - f_s)]$ represents the preference on the set of acts $\mathcal{F}_s$.

Take any two distinct states $s, s' \in \mathcal{S}$ and consider the restriction of $\succcurlyeq$ on the set of acts $\mathcal{F}_{\{s,s'\}} := \mathcal{F}_s \cap \mathcal{F}_{s'}$. On this set both W_s and $W_{s'}$ represent $\succcurlyeq$. Uniqueness results imply that $\phi_{-s} = \phi_{-s'}$ and $u_{-s'} = u_{-s}$ (as both have v in common) and that $\rho_{-s,t} = \rho_{-s',t} =: \rho_t$ for all $t \in \mathcal{S} \setminus \{s, s'\}$. As $s, s' \in \mathcal{S}$ were chosen arbitrary it follows that the positive numbers $\rho_{-s,t}$ are independent of $s \in \mathcal{S}$, such that we have n positive numbers ρ_t, $t \in \mathcal{S}$. Similarly this holds for ϕ_{-s} and u_{-s}; we write $\tilde{\phi}$ and $\tilde{u}$, respectively, instead. Further, continuity at 0 implies $\tilde{u}(0) = 0$ and, hence, $\tilde{\phi} : \mathbb{R}_+ \to \mathbb{R}$ with $\tilde{\phi}(0) = 0$.

Now we set $\rho = \sum_{t \in \mathcal{S}} \rho_t$ and define

$$\pi_t := \frac{\rho_t}{\rho} \text{ for each } t \in \mathcal{S}$$

and obtain a subjective probabilities for the states in $\mathcal{S}$. Finally, we define $u := \rho\tilde{u}$ and adjust $\tilde{\phi}$ as $\phi := \tilde{\phi}[\frac{(\cdot)}{\rho}]$ to maintain the same domain for $\tilde{\phi}\circ\tilde{u}$ for $\phi\circ u$.

Hence, we have shown that the representations W of $\succcurlyeq$ on $\mathcal{F}$ is of the form

$$W(f) = v(f_s) + \phi[\sum_{t\in\mathcal{S}\setminus\{s\}} \pi_t u(f_t - f_s)] \text{ for } f \in \mathcal{F}_s, s \in \mathcal{S}.$$

Now we apply again tradeoff consistency for chances to show that ϕ is linear. As all functions in the preceding representation are continuous, it follows that, locally, we can always find outcomes $y \neq x$ and y', x', and nonnegative c, d, c' such that the following three equations hold:

$$v(y) + \phi[\sum_{t\in\mathcal{S}\setminus\{1\}} \pi_t u(c)] = v(x) + \phi[\sum_{t\in\mathcal{S}\setminus\{1\}} \pi_t u(d)],$$

$$v(y) + \phi[\sum_{t\in\mathcal{S}\setminus\{1\}} \pi_t u(d)] = v(x) + \phi[\sum_{t\in\mathcal{S}\setminus\{1\}} \pi_t u(c')]$$

and

$$v(y') + \phi[\sum_{t\in\mathcal{S}\setminus\{1,2\}} \pi_t u(c)] = v(x') + \phi[\sum_{t\in\mathcal{S}\setminus\{1,2\}} \pi_t u(d)]$$

hold. As $y \neq x$ it follows that c, d, c' are distinct chance outcomes and, hence, $y' \neq x'$. Written in terms of preferences, the last three equations are equivalent to

$$(y, y + c, \dots, y + c) \sim (x, x + d, \dots, x + d),$$
$$(y, y + d, \dots, y + d) \sim (x, x + c', \dots, x + c')$$

and

$$(y', y', y' + c, \dots, y' + c) \sim (x', x', x' + d, \dots, x' + d)$$

respectively. Applying tradeoff consistency for chances means that the following indifference also holds

$$(y', y', y' + d, \dots, y' + d) \sim (x', x', x' + c', \dots, x' + c')$$

or, by substituting the representation W, equivalently, that the next equality is satisfied

$$v(y') + \phi[\sum_{t\in\mathcal{S}\setminus\{1,2\}} \pi_t u(d)] = v(x') + \phi[\sum_{t\in\mathcal{S}\setminus\{1,2\}} \pi_t u(c')].$$

Combining the four preceding equations by taking differences, we obtain

$$\phi[\sum_{t\in\mathcal{S}\setminus\{1\}} \pi_t u(c)] - \phi[\sum_{t\in\mathcal{S}\setminus\{1\}} \pi_t u(d)] = \phi[\sum_{t\in\mathcal{S}\setminus\{1\}} \pi_t u(d)] - \phi[\sum_{t\in\mathcal{S}\setminus\{1\}} \pi_t u(c')],$$

$$\phi[\sum_{t\in\mathcal{S}\setminus\{1,2\}} \pi_t u(c)] - \phi[\sum_{t\in\mathcal{S}\setminus\{1,2\}} \pi_t u(d)] = \phi[\sum_{t\in\mathcal{S}\setminus\{1,2\}} \pi_t u(d)] - \phi[\sum_{t\in\mathcal{S}\setminus\{1,2\}} \pi_t u(c')].$$

By setting $\pi_{\mathcal{S}\setminus\{1\}} := \sum_{t\in\mathcal{S}\setminus\{1\}}\pi_t > 0$ and $\pi_{\mathcal{S}\setminus\{1,2\}} := \sum_{t\in\mathcal{S}\setminus\{1,2\}}\pi_t > 0$, and noting that $\pi_{\mathcal{S}\setminus\{1\}} \neq \pi_{\mathcal{S}\setminus\{1,2\}}$ we obtain

$$\phi[\pi_{\mathcal{S}\setminus\{1\}}u(c)] - \phi[\pi_{\mathcal{S}\setminus\{1\}}u(d)] = \phi[\pi_{\mathcal{S}\setminus\{1\}}u(d)] - \phi[\pi_{\mathcal{S}\setminus\{1\}}u(c')],$$

$$\phi[\pi_{\mathcal{S}\setminus\{1,2\}}u(c)] - \phi[\pi_{\mathcal{S}\setminus\{1,2\}}u(d)] = \phi[\pi_{\mathcal{S}\setminus\{1,2\}}u(d)] - \phi[\pi_{\mathcal{S}\setminus\{1,2\}}u(c')].$$

This means that, locally, for all chances c, d, c' whenever one of the preceding equation hold then also the other equation is satisfied. This implies, that locally $\phi[\pi_{\mathcal{S}\setminus\{1\}}u(\cdot)]$ and $\phi[\pi_{\mathcal{S}\setminus\{1,2\}}u(\cdot)]$ are proportional. As $\pi_{\mathcal{S}\setminus\{1\}} \neq \pi_{\mathcal{S}\setminus\{1,2\}}$ and both probabilities are positive, this proportionality implies, locally, that ϕ is linear. As all functions are continuous, local linearity implies global linearity. Thus, our representation W of $\succcurlyeq$ on $\mathcal{F}$ can be written in the form

$$W(f) = v(f_s) + \sum_{t\in\mathcal{S}\setminus\{s\}} \pi_t u(f_t - f_s) \text{ for } f \in \mathcal{F}_s, s \in \mathcal{S}.$$

Hence, we have derived statement (i) of the theorem.

By construction the probabilities $\pi_s, s \in \mathcal{S}$ are uniquely determined. By construction, for positive α and some constant β, we can replace v and u by $\alpha v + \beta$ and αu, respectively. That there is no further flexibility in the choice of these functions follows from Lemma 6. This concludes the proof of Theorem 7. □

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