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by Thomas S. Lontzek, Daiju Narita

No. 1539 | August 2009

Web: www.ifw-kiel.de

Kiel Institute for the World Economy, Düsternbrooker Weg 120, 24105 Kiel, Germany

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The effect of uncertainty on decision making about climate change mitigation. A numerical approach of stochastic control*

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Abstract: We apply standardized numerical techniques of stochastic optimization (Judd [1998]) to the climate change issue. The model captures the feature that the effects of uncertainty are different with different levels of agent's risk aversion. A major finding is that the effects of stochasticity differ even in sign as to emission control with varying parameters: introduction of stochasticity may increase or decrease emission control depending on parameter settings, in other words, uncertainties of climatic trends may induce people's precautionary emission reduction but also may drive away money from abatement.

Keywords: climate change and uncertainties, stochastic control, climate policy

JEL classification: C63, Q54, D81

Thomas S. Lontzek
Kiel Institute for the World Economy
24100 Kiel, Germany
Telephone: +49-431-8814-405
E-mail: thomas.lontzek@ifw-kiel.de

Daiju Narita
Kiel Institute for the World Economy
24100 Kiel, Germany
Telephone: +49-431-8814-212
E-mail: daiju.narita@ifw-kiel.de

*We would like to thank Ken Judd, Gernot Klepper, Sonja Peterson and Till Requate for their helpful suggestions and review of earlier drafts. Ole Wilms provided excellent research support. We greatfully acknowledge the financial support provided by the German Research Foundation (the "Future Ocean" Cluster of Excellence program) for one of the authors (D.N.).

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1 Introduction

Uncertainty has been regarded as a key issue in the economics of climate change (for reviews, see Heal and Kriström [2002], Peterson [2006] and Pindyck [2007]). While the entire range of research on climate change and uncertainty goes well beyond the realm of economics, a question with a particular economic implication regarding this topic is the decision making under uncertainty whose outcome is not reversible. The concept of quasi-option value (Arrow and Fisher [1974] and Henry [1974]) clarifies that the combination of irreversibility and uncertainty, which is the case for climate change being caused by irreversible accumulation of carbon dioxide in the atmosphere, would justify precautionary actions against the worst possible outcome, in other words, stronger mitigation under uncertainty relative to a deterministic case.

There are a number of studies that examine the validity and applicability of this thesis. They are broadly categorized into two groups. The first group, whose works draw on Epstein's seminal paper (Epstein [1980]), is analytical models with simple settings (often limited to two time periods) to clarify the conditions in which uncertainty leads to precautionary actions. A major insight obtained by this set of literature is that the effect of uncertainty becomes ambiguous if two sorts of irreversibility coexist, namely the irreversibility of atmospheric carbon dioxide concentrations and of investment in mitigation that is sunk (Kolstad [1996a], Kolstad [1996b], Pindyck [2000], Fisher and Narain [2003]). If a part of investment costs in mitigation is not recoverable, a wait-and-see approach to delay actions may rather be preferred because of a possibility that climate change proves to cause smaller damage than expected, in other words, overinvestment in mitigation becomes evident. Other papers in this group (e.g. Ulph and Ulph [1997], Gollier et al. [2000]) look into some other mathematical features leading to the result, such as sufficiency of conditions, the effects of functional shape, and informational structures.

The second group of studies addressing the above question is of integrated assessment models incorporating uncertainty, whose examples are Peck and Teisberg [1993], Nordhaus [1994], Nordhaus [2008], and Pizer [1999]. They use comprehensive economic-climate models calibrated with empirical data on key parameters (e.g., TFP, climate sensitivity to carbon dioxide increase, discount rate) showing a variance of estimates. Here, the effects of uncertainties are examined essentially through a large number of runs with parameters being set at different levels. This group of studies generally show that uncertainty leads to stronger mitigation, although apparently what matters most in their models is not the uncertainty of climate system but of growth and technology parameters such as TFP.

While the two groups of works shed light on the question in a considerable way, there is still an unfilled gap between the literatures. On the one hand, analytical models are only solvable with parsimonious assumptions, and a number of parameters commonly considered in modeling climate change are omitted. On the other hand, the second group of studies, integrated assessment models, does not directly conduct stochastic optimization due to computational difficulties. This means that they do not take account of the effect that uncertainty influences optimal decisions through agents' risk aversion. Furthermore, this limitation frames constraints on uncertainties they investigate; they mostly focus on uncertainties of parameters (e.g., energy intensity) with pre-defined probability distributions, not randomness of state or control variables themselves (e.g., atmospheric temperatures). This feature makes the models unattractive in examining the question of uncertainty and irreversibility about climate, because a part of climatic patterns could only be explained by highly non-linear, possibly inherently unpredictable, mechanisms of the climate system, whose evidence includes paleoclimatic records of abrupt temperature changes (e.g., NRC [2002]). Accordingly, studies of integrated assessment models have not explicitly examined this question.

This paper is an attempt to fill the current gap between the two sets of scholarship described above. We directly perform stochastic optimization with variable randomness represented as a Brownian motion. A numerical approach allows us much greater latitude for parameter choice than analytical model studies would do. Stochastic dynamic optimization has an established body of analytical model studies (in the field of environmental and resource economics, e.g., Arrow and Chang [1982], Tsur and Zemel [1998]), but has been generally considered difficult in finding numerical solutions. Recently, however, standardized techniques are developed (e.g. Judd [1998]), and some simple models are now able to be solved readily. Our approach is to apply these techniques to the climate change issue with representations that are simple but could still have direct relevance to the actual climatic-economic interactions. Though not the focus of this paper, this approach would leave us a scope to link the quasi-option value literature and the economic studies on abrupt climate change (e.g. Azar and Lindgren [1992], Keller et al. [2004], Lempert et al. [2006], McInerney and Keller [2008] and Weitzman [2009]). Our research question here is the effect of climatic uncertainty on the optimal mitigation policy. Our analysis covers a large range of the parameter space, in particular the degree of risk aversion and the level of uncertainty. We identify regions of the state space for which higher levels of uncertainty or risk aversion result in different policy rules for emission control. Similarly, given a certain state of the world we conclude that the effect of uncertainty on emission control changes (in level and sign) with the degree of risk aversion.

We proceed as follows: In section 2 we describe the model framework. In section 3 we briefly describe the Chebyshev collocation method, the computational technique which we use for solving our stochastic control problem in continuous time. Section 4 presents the main results of our model and provides a discussion. Section 5 concludes

2 The Model

Consider an economy where total output Y is a function of the capital stock K, with $Y_K > 0$ and $Y_{KK} < 0$. The production process generates emissions $\epsilon \cdot Y$, where ϵ denotes the emissions coefficient of output. With additional expenditure, the amount of emissions is reduced; m represents the fraction of carbon emissions which is under control, i.e. not emitted in the atmosphere. Consequently, the atmospheric stock of carbon S evolves with

$$dS = \epsilon \cdot Y(K) \cdot (1 - m) - \beta \cdot S \qquad (1)$$

where β is the constant removal rate of atmospheric carbon into the ocean. At this point we assume that the atmospheric stock of carbon causes a rise in the level of global mean temperature. Let $T(S)$ be the increase of global mean temperature from the pre-industrial level with $T_S > 0$ and $T_{SS} \geq 0$. We assume that rising levels of global mean temperature cause damage to output and the damage is subject to randomness. Denote the damage by $D(T,\eta)$ with η being a scaling factor of the temperature's impact on damage: we assume $D_T, D_\eta > 0$, $D_{TT}, D_{\eta\eta} > 0$, $D_{T,\eta} > 0$ and $D(T,0) = D(0,\eta) = 1$. For the rest of the paper we assume that η is stochastic with

$$d\eta \quad = \quad \theta \cdot (\bar{\eta} - \eta) + \sigma dB \tag{2}$$

i.e, the damage coefficient follows an Ornstein-Uhlenbeck process, the continuous time equivalent of a mean-reverting AR(1) process. The mean of η is denoted by $\bar{\eta}$ and θ is the strength of mean reversion. For the diffusion, we assume $B \sim (0, \sigma^2)$. Furthermore, the output balance condition reads

$$\frac{Y(K)}{D(T(S),\eta)} \quad = \quad I + c + M(m) \tag{3}$$

The left-hand side of (3) is the net output inclusive of damage. The net output is in balance with the sum of the following: (i) consumption c which yields utility $U(c)$ with $U_c > 0$ and $U_{cc} < 0$; (ii) $M(m, Y(K))$, the emission control costs with $M_m > 0$, $M_{mm} > 0$, $M_K > 0$, $M_{mK} > 0$ and $M_{KK} = 0$; (iii) capital accumulation via investment I. The stock of capital K evolves according to

$$dK \quad = \quad I - \delta \cdot K \tag{4}$$

where δ is the capital depreciation rate. Our purpose is to investigate the dynamically optimal choice of consumption, emissions control and capital investment given uncertainty about the temperature's impact on damage to gross output. To this end, we formulate the problem from the social planner's perspective. Given the uncertainty over η, the social planner maximizes the expected present value welfare.

$$\max_{c_t > 0, 0 \leq m_t \leq 1} E \int_0^\infty e^{-\rho t} \left[U(c_t) \right] dt \tag{5}$$

subject to (1)-(4) and $S(0) = S_0$, $K(0) = K_0$ and $\eta(0) = \eta_0$. To solve (5) we perform stochastic control, the continuous time version of dynamic programming. The corresponding Hamilton-Jacobi-Bellman (HJB) equation is [1]

$$
\begin{aligned}
0 = \max_{c>0, 0\leq m\leq 1} \{U(c) &+ V_S(S,K,\eta)(\epsilon\cdot Y(K)\cdot(1-m) - \beta\cdot S) \\
&+ V_K(S,K,\eta)(\frac{Y(K)}{D(\eta,S))} - c - M(m,K) - \delta\cdot K) \\
&+ V_\eta(S,K,\eta)(\theta\cdot(\bar{\eta}-\eta)) \\
&+ \frac{1}{2}\sigma^2 V_{\eta\eta}(S,K,\eta) - \rho V(S,K,\eta)\}
\end{aligned} \tag{6}
$$

where $V(S,K,\eta)$ is the value function. A solution to (6) requires finding a value function and policy functions $c(S,K,\eta)$ and $m(S,K,\eta)$ which constitute explicit control rules. The first-order conditions for c and m are

$$U_c = V_K(S,K,\eta) \tag{7}$$

$$M_m = -\frac{V_S(S,K,\eta)\cdot\epsilon\cdot Y(K)}{V_K(S,K,\eta)} \tag{8}$$

Equation (7) states that the marginal utility from consumption should be equal to the derivative of the value function with respect to capital, i.e. the shadow price of capital. From (8) it can be easily seen that $V_S \geq 0$. The optimal choice of m, the emissions control rate, thus positively depends on the shadow price of atmospheric carbon (in absolute terms) and instant emissions. It negatively depends on the shadow price of capital.

A closed form solution to (6)-(8) could be obtained by applying specific function forms to Y, D, M, T and U and using an intelligent guess for the value function $V(S,K,\eta)$. However, due to the dimension of the state space and the nonlinearities of the functional forms we are not able to derive a closed form solution. Instead, we determine the value function and the policy functions numerically.

[1] Notice that by setting up the maximization problem as in (6), we do not restrict capital investments I to be positive. In fact, for some areas of the state and parameter space optimal investment is negative.

3 The Approximation Method

From the first-order conditions (7) and (8) we can obtain explicit solutions for the optimal stochastic control of c and m as functions of the state variables.

$$\tilde{c} = \Gamma_U^{-1}(V_K(S, K, \eta)) \tag{9}$$

$$\tilde{m} = \Gamma_M^{-1}\left(-\frac{V_S(S, K, \eta) \cdot \epsilon \cdot Y(K)}{V_K(S, K, \eta)}\right) \tag{10}$$

Inserting (9) and (10) into (6) we obtain the concentrated HJB equation in terms of the value function and its derivatives with respect to the states. Thus, the concentrated HJB equation is three-dimensional in S, K and η and reads

$$\begin{aligned} 0 &= V_S(S, K, \eta)(\epsilon \cdot Y(K) \cdot \left(1 - \Gamma_M^{-1}\left(-\frac{V_S(S, K, \eta) \cdot \epsilon \cdot Y(K)}{V_K(S, K, \eta)}\right)\right) - \beta \cdot S) \\ &+ V_K(S, K, \eta)(1 + \frac{Y(K)}{D(\eta, S))} - \Gamma_U^{-1}(V_K(S, K, \eta)) + \frac{V_S(S, K, \eta) \cdot \epsilon \cdot Y(K)}{V_K(S, K, \eta)} - \delta \cdot K) \\ &+ V_\eta(S, K, \eta)(\theta \cdot (\bar{\eta} - \eta)) + \frac{1}{2}\sigma^2 V_{\eta\eta}(S, K, \eta) - \rho V(S, K, \eta) \end{aligned} \tag{11}$$

Equation (11) constitutes a nonlinear second-order partial differential equation which can be solved numerically using projection methods (Judd [1992][1998]). Projection methods work very well with continuous-time, continuous-state problems (Judd [1998]). We estimate the value function with the Chebyshev collocation method using Matlab's CompEcon toolbox (Miranda and Fackler [2002]). Making use of the Weierstrass theorem, the collocation method approximates the solution to (11) with a linear combination of basis functions whose coefficients approximately solve (11) at specific collocation nodes by value function iteration with Newton's method until a convergence rule is satisfied. The approximated value function is given by

$$\tilde{V}(S, K, \eta) = \sum_{i=1}^{n_i} \sum_{j=1}^{n_j} \sum_{k=1}^{n_k} g_{ijk} T_i(x_S) T_j(x_K) T_k(x_\eta)$$

$T_i(x_S), T_j(x_K)$ and $T_k(x_\eta)$ are n_i, n_j, n_k-degree Chebyshev polynomials which are evaluated at the states with x_S, x_K, x_η being the mapping $[S_{\min}, S_{\max}] \times [K_{\min}, K_{\max}] \times [\eta_{\min}, \eta_{\max}] \longmapsto [-1, 1] \times [-1, 1] \times [-1, 1]$. The collocation coefficients g_{ijk} are then estimated in order to deliver a good approximation of (11).

4 Results and Discussion

The functional forms and parameter values used for the numerical analysis are reported in the Appendix. With these parameter values we compute numerically the deterministic steady state and obtain $\tilde{S} = 1546.6$, $\tilde{K} = 1180.2$. Furthermore, we define $\eta = 1$ in the deterministic case. Given these values we set up the projection grid by discretizing the spate space around the steady state. We choose $S \in [800, 3500]$, $K \in [500, 3000]$ and $\eta \in [0, 2]$. The Chebyshev polynomials are of degree 10 in all states i.e.: $n_i, n_j, n_k = 10$. Figure 1 illustrates the value function for the stochastic

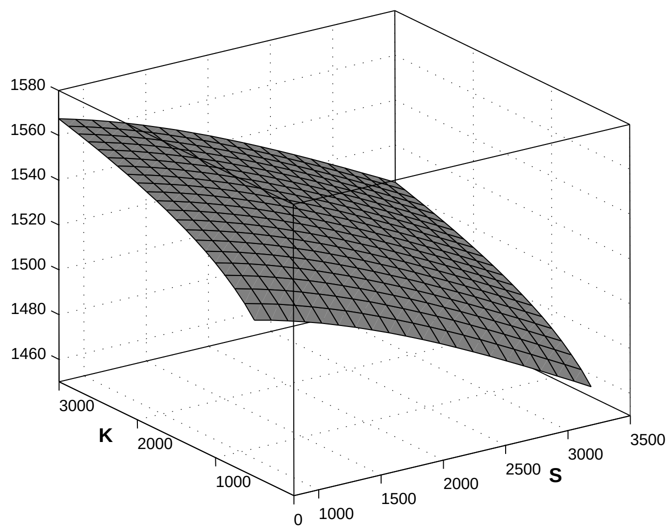

Figure 1: Value function

case in the $S - K$ grid [2]. The value function is concave and smooth. It increases with larger volumes of the capital stock and decreases with rising atmospheric carbon concentrations[3]. The relative value function residual is at around $\times 10^{-8}$ over the entire

[2]For the graphical presentation of the results we choose $\eta = 1$ unless stated otherwise

[3]Notice that low levels of the capital stock imply low levels of gross output. This in turn results in low emissions. On the other hand, lower output volumes are available for consumption, investment

state grid. Figure 2 displays the shadow values of the atmospheric carbon stock (V_S)

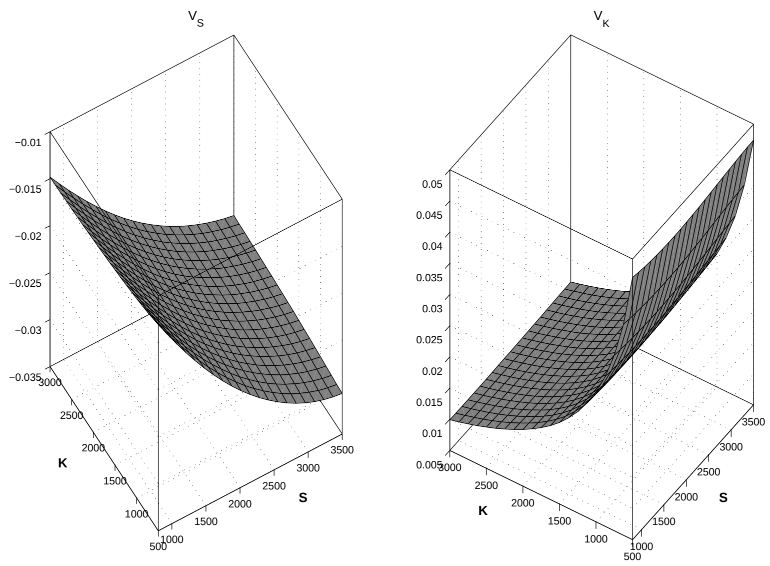

Figure 2: Shadow prices of atmospheric carbon stock (λ_S) and capital stock (λ_K)

and the capital stock (V_K). Notice that V_S is negative over the entire state space - an intuitive result, since rising temperature levels are proportional to the atmospheric carbon stock. This fact also explains why V_S decreases with rising levels of the carbon stock while it is rather invariant to changes in capital. An analogous picture is obtained for V_K, the shadow value of the capital stock (right plot in Figure 3). It is positive over the entire state space and decreases with rising levels of the capital stock. Figure 3 maps the policy functions for consumption c, emissions control m and investment I into the $K - S$ space, again for the stochastic case. The optimal consumption policy rule follows the Euler equation which sets equal marginal utility to the shadow price of capital. Consumption thus increases with the level of capital.

The emissions control policy generally replicates the tendency that most integrated assessment models exhibit, i.e., both carbon stock and capital accumulation increase enhances mitigation (e.g., Nordhaus, 1994). Notice that for a constant level of K, a higher atmospheric carbon concentration generates more damage to output, and that less output is available to be divided between consumption, emission control and investment. Also, for any level of K the emissions control is larger increases with

and emission control. Furthermore, for any level of capital a higher S invokes more damage and consequently less net output while the level of gross output is unchanged.

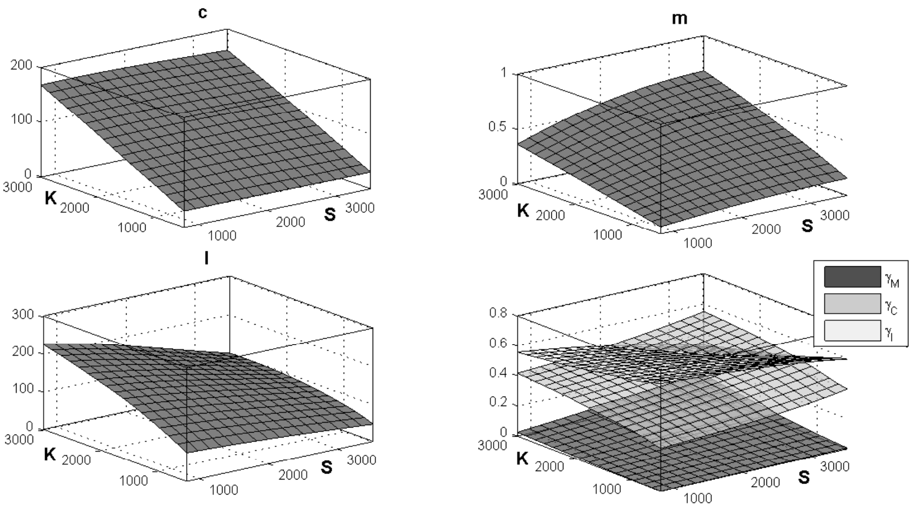

Figure 3: Policy functions for consumption (c), emission control (m), investment(I) and shares of net output spent on c, m and I

larger values of S with a constant consumption and therefore capital investments must decrease in order to balance the economy's budget (Equation 3). This behavior is shown in the lower left plot of Figure 3. On the other hand, higher levels of the capital stock invoke more investment. The lower right plot in Figure 3 displays the shares of net output[4] spent on consumption, emissions control and investment which we define as γ_M, γ_C and γ_I respectively[5] We observe that γ_C and γ_M both follow the same pattern. They increase with higher levels of capital and atmospheric carbon stock. However, while the share of net output spent on consumption ranges from 25% (low K, low S) - 60% (low K, low S), much less fractions of net output are used for emission control. Its share ranges from 5% (low K, low S) - 20% (low K, low S). On the contrary, the policy function for investment implies lower investment values for rising levels of capital and atmospheric carbon stock.

In order to shed more light on the effect of uncertainty on the distribution of net output over consumption, emission control and investment Figure 4 displays the absolute change in γ_C, γ_M and γ_I when including uncertainty. Three important points can be made: 1) For low values of the atmospheric carbon stock uncertainty leads to higher emission control and consumption while it lowers capital accumulation. 2)

[4]Net output is defined as $Y^{net} = \frac{Y}{D}$

[5]Consumption, emission control and investment are defined in units of output. It holds that: $\gamma_M + \gamma_C + \gamma_I = 1$

The previous effect is reversed for high values of the atmospheric carbon stock. When uncertainty is included, a larger share of net output is spent on capital accumulation atmospheric while a lower share of net output is used for consumption and emission control. 3) The impact of uncertainty on changes in the output shares used optimally for c, m and I decreases (lower amplitude) with smaller levels of the capital stock. The latter effect mirrors the fact that a low value of the capital stock limits the freedom of action to adapt to stochasticity. From Figure 4 it becomes clear that if the carbon

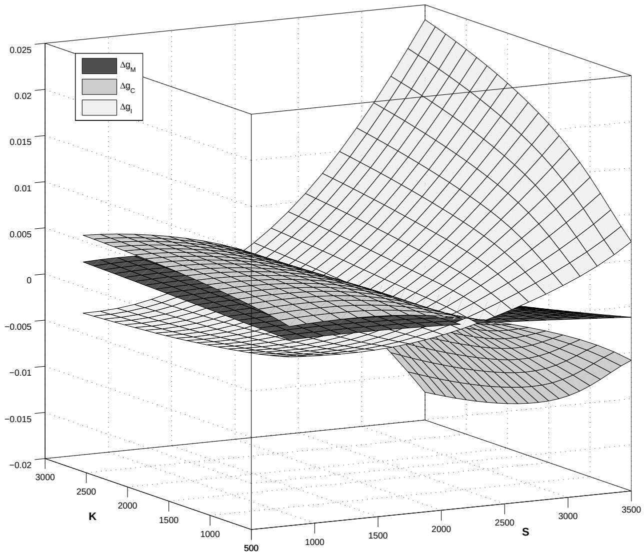

Figure 4: Difference in the shares of net output spent on emission control, Investment and consumption when uncertainty is included $(\Delta g_M, \Delta g_I, \Delta g_C)$ respectively

content in the atmosphere is large, uncertainty about damage to output induces a reallocation of net output towards capital services. A perhaps striking feature of this model is also that in the latter case emission control is reduced. To obtain more insights on the effect of stochasticity on emission control, we examine the optimal levels of emission control with varying levels of risk aversion and randomness (Figure 5). We show nine contour plots in the $\alpha - \sigma$ space for different levels of S and K. They exhibit three major patterns. (1) The general tendency is that both a high capital and a high carbon stock results in a high emission control. A large carbon stock corresponds to a large negative carbon price and a large emission reduction, while a large capital stock is linked to a low capital return and thus a diversion of resource from investment to emission control. (2) The risk aversion is a very influential

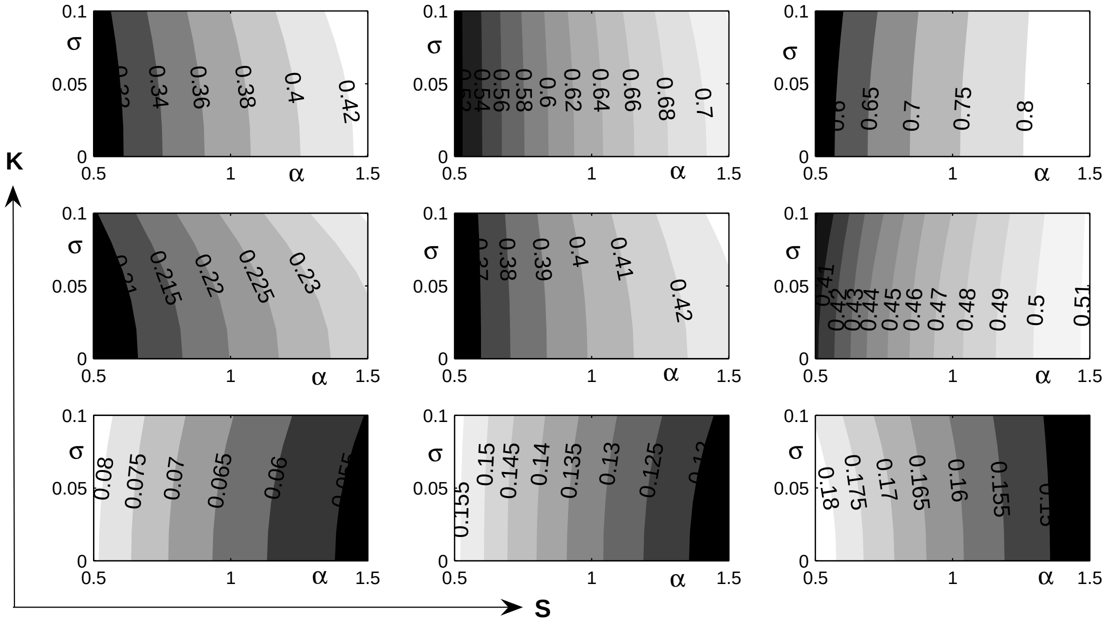

Figure 5: Optimal Emission control - Contour plots in the $\alpha - \sigma$ space for combinations of S and K. $S \in [800, 2150, 3500]$ and $K \in [500, 1750, 3000]$, with [S,K]=[800,500] in the lower left subplot.

parameter on the level of emission control. Interestingly, however, the risk aversion exerts different effects on the control level depending on the level of capital. With the middle or high capital, more risk aversion leads to more abatement. Meanwhile, the emission control decreases with higher levels of risk aversion when the capital level is low. This is because a risk conscious agent prefers capital investment over emission control when the return to capital is relatively high (i.e., low capital), in other words, capital investment facilitates intertemporal income smoothing more effectively than emission control does. (3) The level of carbon stock has a critical meaning for the effect of stochasticity on the emission control. Higher uncertainty leads to higher emission control with a low S but lower emission reduction with a high S. In other words, a risk conscious agent rather prefers consumption over emission control when climate mitigation needs considerable effort and in turn the effect of actions is highly uncertain in absolute terms.

Among the above findings, the point (3), the ambiguity regarding the effect of uncertainty on optimal emission control levels, addresses a feature that is not adequately discussed in previous studies in the economics of climate change. In fact, this am-

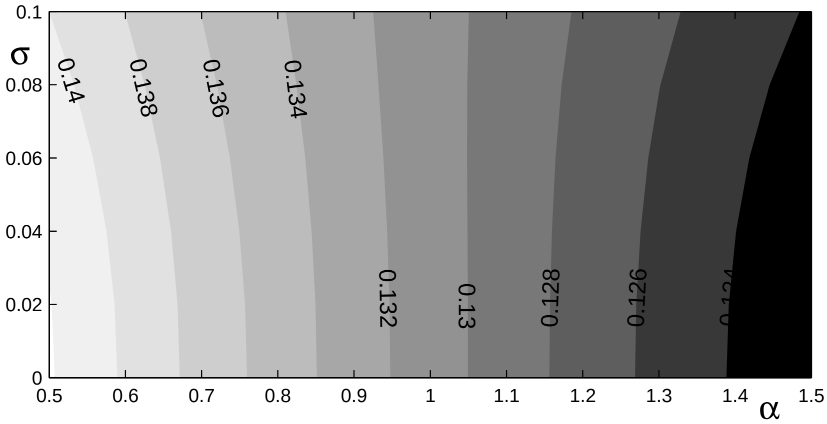

Figure 6: Optimal Emission control - Contour plot in the $\alpha - \sigma$ space for high S and low K ([S,K]=[3500,500]).

biguity is a persistent characteristic in our model results, and the model can present it even in a more illustrative way. Figure 6 is a contour plot for low K and high S when the climate change damage coefficient is set low ($\eta = 0.5$). It clearly shows that the effect of uncertainty is even dependent on the level of risk aversion. With a low risk aversion, uncertainty decreases emission reduction, whereas it increases emission reduction with a high risk aversion.

A particularly interesting point regarding these patterns is that uncertainty may in fact reduce the optimal level of emission control. This is against the most basic argument of quasi-option value, but it is straightforward to interpret the feature. Previous studies already clarified that if investment in abatement involves sunk costs, uncertainty in stock pollution can either enhance or decrease abatement, dependent on the parameter choice (e.g., Pindyck [2000]). This is because both the installment of abatement equipment and the pollution stock have irreversibility, and the effect of uncertainty only arises in balance of those two opposing factors. Our model does not have an explicit representation of sunk investment on abatement, but there is a similar, though indirect, mechanism at work. Abatement costs (flow) are subtracted from the output, and thus their increase reduces either consumption, capital investment, or both, if the output is unchanged. Capital produces a continuous flow of income from the time of investment onwards, and forgone capital investment due to excessive abatement therefore sets irreversibility in the other direction. This argument could be paraphrased as follows: The standard argument of quasi-option value says that the presence of uncertainty leads to increasing abatement of stock pollution because one cannot reduce the pollution stock later in case the pollution damage is greater than expected. In our settings, a similar argument holds for the other direction. If we overspend our resource on abatement, capital investment could be comparatively decreased. Lower investment leads to lower capital accumulation. By the time we realize the overspending on abatement, the accumulated abatement cannot be converted into capital, and one cannot recover the income flow that capital would bring about if our resource was allocated in investment, not abatement. Relative significance of capital return and climate damage determines the dynamics to either of the two directions.

5 Conclusion

We carried out a numerical stochastic optimization in the context of climate change. We applied standardized numerical techniques of stochastic optimization recently developed Judd [1998] to the climate change issue, with an assumption of stochasticity in the climate system. The novelty of this study is that we directly performed stochastic dynamic optimization, rather than reproducing randomness by conducting a large number of simulation runs, to see changes of key determinants of climate policy. An advantage of our stochastic optimization approach over previous climate-economy simulation studies is that the model internalizes agents preference about risk in optimization. Our analysis covers a large range of the parameter space, in particular the degree of risk aversion and the level of uncertainty. We identify regions of the state space for which higher levels of uncertainty or risk aversion result in different policy rules for emission control.

The results show that the effects of uncertainty are indeed different with different levels of agents risk aversion. A main finding is that with the effects of stochasticity differ even in sign as to emission control with varying parameters: introduction of stochasticity may increase or decrease emission control depending on parameter settings, in other words, uncertainties of climatic trends may induce peoples precautionary emission reduction but also may drive away money from abatement. This paper's conclusions would set a call for a more precise conceptualization about the meaning of uncertainties in the decision making on climate change. This aspect would have a particular importance in the context of policy discussions, where uncertainty is often used as a justification for policy actions yet tends to be vaguely framed, notably as phrased in the United Nations Framework Convention on Climate Change's objective to "prevent dangerous anthropogenic interference with the climate system." Finally, while our model does highlight some important features of uncertainties and climate change, the simulations are admittedly simplistic for explaining the complex phenomena of climate change. A more comprehensive numerical stochastic model, perhaps with uncertainties in technological change and global business cycle in addition to climate indicators, would allow us to conduct a complete sensitivity analysis of parameters. Impacts of uncertainties about large discrete shocks, a feature that could be represented with a jump process, should also be a future research question.

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6 Appendix

We apply the following functional forms. with $A, \nu, \kappa, \tau, \epsilon, \psi, S_{PI}, \alpha > 0$

$$Y(K) = A \cdot K^{\nu} \tag{12}$$

$$D(\eta, T(S)) = 1 + \kappa \cdot (\eta \cdot T(S))^2 \tag{13}$$

$$T(S) = \tau \cdot (S - S_{PI}) \tag{14}$$

$$M(m) = \psi \cdot \epsilon \cdot Y \cdot m^2 \tag{15}$$

$$U(c) = \frac{c^{1-\alpha}}{1-\alpha} \tag{16}$$

Concerning the parameter space, we use the following specification:

Parameter	Value	Parameter	Value
ν	0.75	S_{PI}	400
κ	0.005	α	0.9
τ	0.003	ρ	0.01
ψ	40	A	1
ϵ	0.1	$\bar{\eta}$	1
θ	0.1	σ	0.05

Table 1: Parameter Values