



*Kiel*

## **Working Papers**

**Kiel Institute  
for the World Economy**



**Hubs and resilience: towards more  
realistic models of the interbank  
markets**

**by Mattia Montagna and Thomas Lux**

**No. 1826 | 02/2013**

**Web: [www.ifw-kiel.de](http://www.ifw-kiel.de)**

Kiel Working Paper No. 1826 | 02/2013

## **Hubs and resilience: towards more realistic models of the interbank markets**

Mattia Montagna and Thomas Lux

**Abstract:** This paper uses a toy financial system to study systemic risk in scale-free interbank networks. Networks are produced according to a *fitness* algorithm, combined with a representation of the balance sheets of the banks. Our generating processes for interbank networks are designed in a way to reproduce the frequently documented features of disassortative behavior, power laws in the degree distributions and power laws in the distribution of bank sizes. The results show the presence of a particular shell structure affecting the spread of an endogenous shock.

**Keywords:** Interbank market, contagion, networks, financial stability.

**JEL classification:** G21, G01, E42

Prof. Dr. Thomas Lux  
Kiel Institute for the World Economy  
Telephone: +49 431 8814 278  
Email: [thomas.lux@ifw-kiel.de](mailto:thomas.lux@ifw-kiel.de)  
24100 Kiel, Germany

Mattia Montagna  
Kiel Institute for the World Economy  
Telephone: +49 431 8814 251  
Email: [mattia.montagna@ifw-kiel.de](mailto:mattia.montagna@ifw-kiel.de)  
24100 Kiel, Germany

Christian-Albrechts-University Kiel  
Department of Economics  
Chair of Monetary Economics and  
International Finance  
24098 Kiel, Germany

Bank of Spain chair of Computational Finance  
Department of Economics, University Jaume I  
Castellón, Spain

---

*The responsibility for the contents of the working papers rests with the author, not the Institute. Since working papers are of a preliminary nature, it may be useful to contact the author of a particular working paper about results or caveats before referring to, or quoting, a paper. Any comments on working papers should be sent directly to the author.*

*Coverphoto: uni\_com on photocase.com*

# Hubs and resilience: towards more realistic models of the interbank markets

MATTIA MONTAGNA\*, THOMAS LUX†

## Abstract

This paper uses a toy financial system to study systemic risk in scale-free interbank networks. Networks are produced according to a *fitness* algorithm, combined with a representation of the balance sheets of the banks. Our generating processes for interbank networks are designed in a way to reproduce the frequently documented features of disassortative behavior, power laws in the degree distributions and power laws in the distribution of bank sizes. The results show the presence of a particular shell structure affecting the spread of an endogenous shock.

**Keywords:** Interbank market, contagion, networks, financial stability.

**JEL classification:** G21, G01, E42

## 1 Introduction

In the last decades more and more efforts have been directed to the study of interbank financial data using tools initially developed in the natural sciences, with the aim to shed light on the contagions effects of shocks through interbank linkages. In particular, a better understanding of the link between the topology of a financial system, where an intricate network of financial entities (like banks and hedge funds) are connected together through a complex web of financial instruments, and the stability of the system itself, namely the ability of networks to absorb shocks and adapt the structure in order to maintain efficiency, became a major issue (Battiston et al, 2009). The relevance of the network structure for regulatory reform of the banking sector has been emphasized by Haldane and May (2011), among others.

The risk of a global systemic failure of the whole system is strongly connected to the topological features of the financial network, and it gives

---

\*Department of Economics, University of Kiel and Kiel Institute for the World Economy, Germany, Email: mattia.montagna@ifw-kiel.de

†Department of Economics, University of Kiel and Kiel Institute for the World Economy, Germany & Banco de España Chair in Computational Economics, University Jaume I, Castellón, Spain, Email: lux@bwl.uni-kiel.de.

rise to the crucial concept of systemic risk. Since the pioneering work of Allen and Gale (2000) in which the relevance of the structure of a financial system for its stability has been highlighted, the study of systemic risk using network approaches has attracted the attention of economists and scientists in general. Nier *et al.* (2007), have studied a simple but versatile random network structure, where the nodes of the networks represent banks and the edges represent interbank liabilities, combined with a representation of the balance sheets of banks. They show how the resilience of the whole system to idiosyncratic shocks is affected by the topological features of the system, such as the connectivity of the nodes. May and Arinaminphaty (2009) provide an analytical explanation of these results using a mean-field approach, providing more insights into the connections between complexity and stability.

The main aim of this paper is to expand this line of research into the determinants of systemic risk in simulated banking systems. By systemic risk we mean the risk of a whole financial system, in this case a set of banks, to collapse as an aftereffect to the initial default of a single unit or a small cluster. After the default of the first bank, the shock is transmitted through the whole system due to a web of debt relationships. This domino effect may cause the whole system to fail. As in the case considered by Nier *et al.* (2007) and May and Arinaminphaty (2005), our networks are static since the single nodes, namely the banks, are not allowed to change their behavior during the spread of the shock, they just passively absorb the propagation of the losses. We are, therefore, considering a situation in which the spread of the shock through the system is faster than the potential changes of the topological features of the interbank network that would be manifested after the reaction of the banks themselves.

In network theory, if high-degree vertices attach to low-degree ones, the resulting graph is said to display a *disassortative mixing* or *disassortative behavior*. A simple way to identify such a structure consists in studying the distribution of the average degree of the neighbours of the vertices belonging to the network. In the case of disassortative mixing, this distribution should be a decreasing function in the degree of the nodes, as a consequence of the attitude of high-degree vertices to link with low-degree ones, and *vice versa*. Disassortative mixing is a frequent feature of real networks, examples are the internet, the World Wide Web, protein interactions and neural networks (Caldarelli, 2007). Interestingly, also most of the interbank money markets seem to be characterized by disassortative behavior, as documented by Boss *et al.* (2004) for the Austrian interbank market, Soramäki *et al.* (2006) for the US Fedwire Network, Iori *et al.* (2008) for the Italian interbank market, and Imakubu and Soejima (2006) for the Japanese interbank money market. Therefore, it seems important to include the well-established stylized fact of disassortative mixing also in the study of artificial financial networks, since this particular structure could affect the ability of a system to absorb

shocks. Another feature that is often present in real networks is a characteristic power law degree distribution, that produces the so-called *scale-free networks*. Scale-free networks are characterized by the presence of hubs, namely nodes with a degree that is much higher than the mean degree of the other nodes. Therefore, in a scale-free network, there is a high probability that many transactions would take place through one of the high-degree nodes of the network. The presence of such hubs make systems in general more prone to a break-down in case of targeted attacks, as the downside of their high connectivity in terms of the shortest paths between any two nodes belonging to the system. Again, in real interbank money markets scale-free degree distributions have been frequently reported. Examples are Inaoka *et al.* (2004) and Imakubu and Soejima (2006) for the Japanese interbank market, and Boss *et al.* (2004) for the Austrian interbank market, while there exist divergent results for the Italian interbank market (Iori *et al.*, (2008), Lux *et al.* (2012)).

Empirical evidence on the size distribution of bank's balance sheets can be found in, for example, Ennis (2001) and Janicki and Prescott (2006). For the U.S., the banking system is characterized by a large number of small banks and a few large banks, and the size distribution seems to be lognormal with a Pareto-distributed tail. A study on the evolution of the banking system in a European Country can be found in Benito (2008), where the presence of few big hubs in the Spanish banking system is highlighted, and, again, the distribution is highly skewed, and it has become more skewed during the last decades.

We construct a Monte Carlo framework for an interbank market characterized by the above empirical features via what is called a *fitness algorithm* (De Masi *et al.*, 2006). With a particular choice of such a function as a generating mechanism for our network, we can make sure that our artificial banking sector also displays a power law degree distribution, disassortative behavior and heterogeneity in the banks' sizes. In particular, in interbank markets characterized by a power law in the size distribution, the default of a single small or medium-sized bank will not affect the stability of the entire system: as one might expect, the losses are easily absorbed by the banks which have deposits on the liability side of the failing bank's balance sheets, and no domino effect occurs. The situation changes when the initially defaulting bank is one of the hubs of the system. In this case the propagation of the shock proceeds like the propagation of a circular wavefront in the water: starting from an initial node, the shock will hit at the same time nodes that are directly linked to the source. Moreover, each time a new node is hit by a wave, it also will become a source itself, expanding the range of nodes that will potentially be affected by the shock. Those are the kind of network effects we are interested in. Note that the results reported so far in the literature using network approaches in order to study domino effects in interbank markets have mostly used either random network models or

networks constructed from aggregate data via a maximum entropy principle (cf. Upper, 2011, for an overview). Both approaches are very likely to underestimate the extent of a contagious spread of disturbance due to the very homogeneous level of activity and connectivity in such artificial networks. In contrast, the above stylized facts show strong heterogeneity for the levels of activity (size of the balance sheets, as well as the extent of connectivity, namely the degree distribution). In addition, the pronounced negative assortativity is also not covered by random networks or those constructed from entropy principles. Moreover, random networks are characterized by a binomial degree distribution (see, for example, Caldarelli (2007)), and so no major hubs exist in such a system. Using power law degree distributions, the process of propagation of endogenous shocks could bring about different results, and should in principle give a more realistic picture of the underlying phenomena.

In the following, section 2 introduces the generating mechanism for realistic (along certain important dimensions) interbank markets, section 3 provides a summary of the main properties of the networks produced by our model, and section 4 introduces the mechanism for the propagation of the shocks and shows the result from the Monte Carlo simulations. Section 5 finally concludes.

## 2 Generating mechanism for a scale-free banking system

We consider an interbank market (IbM) composed of  $n$  financial entities linked together by their claims on each other. It seems natural to use network theory in order to represent and study such a system: each bank in the IbM will be represented as a node in the network, and the information of the loans among banks will be included in the edges of the network. These edges are directed and weighted, the weight of the link starting from node  $i$  and pointing to node  $j$  being the total amount of money that bank  $i$  lends to bank  $j$ . In order to proceed a modest step towards a more realistic representation on the interbank market, we will construct our toy financial system in a way to represent the documented empirical features highlighted in the introduction. Following Nier *et al* (2007), we use the scheme represented in Figure 1 in order to represent the balance sheets of the bank. The assets  $A_i$  of each bank ( $i = 1, 2, \dots, n$ ) are partitioned into interbank loans  $l_i$  and external assets  $e_i$ :

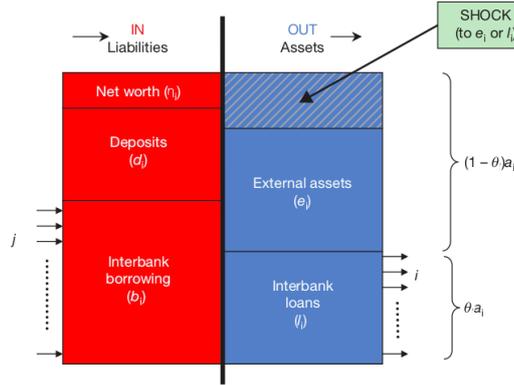
$$A_i = l_i + e_i \quad (1)$$

The liabilities  $I_i$  of each bank are partitioned into the internal borrowing  $b_i$ , customers' deposits  $d_i$ , and the net worth  $\eta_i$ :

$$I_i = b_i + d_i + \eta_i \quad (2)$$

Solvency requires that the difference between a bank's assets and its liabilities be positive, that is:

$$\eta_i \equiv (l_i + e_i) - (d_i + b_i) \geq 0 \quad (3)$$



**Fig. 1:** The balance sheets of each bank  $i$  belonging to the IbM.

If relationship 3 is not fulfilled, bank  $i$  becomes insolvent. Note that we could instead impose a minimal capital requirement and intercept the bank's operations if its capital falls below a threshold. For most purposes that would leave our results qualitatively unchanged as it would just lead to a linear rescaling of the balance sheet.

Following Nier *et al.* (2007), we impose the following relations, that hold for all the banks belonging to the IbM:

$$e_i = \theta A_i \quad (4)$$

$$l_i = (1 - \theta) A_i \quad (5)$$

$$\eta_i = \gamma A_i \quad (6)$$

This enables us to characterize the evolution of the balance sheet of the banks using the common pair of parameters  $\theta$  and  $\gamma$ . Unlike Nier *et al.* who investigate a banking sector with banks of equal size of balance sheets and interbank liabilities, we try to mimic some of the documented dimensions of heterogeneity in the banking sector.

The empirical properties of real interbank networks that we attempt to reproduce are the disassortative behavior and power law in the degree and size distributions. To this end, we arrange the nodes on a scale free network according to the following algorithm:

1. we start with an assumption on the distribution of the size of the banks. Using  $A_i$  as parameter indicating the size of a bank, we assume that  $\rho(A_i) \propto A_i^{-\tau}$  (and  $A_i \in [a, b]$ ) so that the sizes distribution will follow a power law, and in the following we will use  $\tau = 2$ . We note that

since this formalism defines the size distribution over a finite range, the numbers  $a$  and  $b$  defining the absolute range of bank sizes will also be of some relevance.

2. once we have drawn the  $n$ -element set  $\{A_i\}$  i.e. the distribution of the total external assets of the banks, we compute the external assets  $e_i$ , the interbank loans  $l_i$  and the net worth  $\eta_i$ , according to equations 4 - 6.
3. we use now the size parameter  $A_i$  as the *peculiarity* of the node. This basically means that we add interbank liabilities to the system in relation to the sizes of each pair of potential trading partners. In order to build up networks in this way, we use a *probability function*  $P(A_i, A_j)$ : this function provides the probability that a bank  $i$  (characterized by total external assets  $A_i$ ) lends money to bank  $j$  (characterized by total external assets  $A_j$ ). In most real IbMs, a pool of small and medium-sized banks usually lend money to the biggest banks of the system, which in turn redistribute liquidity to external financial markets or within the IBM itself ( Iori *et al.* (2008), Cocco *et al.* (2009), Fricke and Lux (2012)). The choice of an appropriate probability function allows to reproduce those important empirical observations. In the following, we will use the three following alternative probability functions for the generation of links:

$$P_1(A_i, A_j) = \left(\frac{A_i}{A_{max}}\right)^\alpha \cdot \left(\frac{A_j}{A_{max}}\right)^\beta \quad (7)$$

$$P_2(A_i, A_j) = c \cdot (A_i + A_j) \quad (8)$$

$$P_3(A_i, A_j) = \theta(A_i + A_j - z) \quad (9)$$

where  $A_{max}$  denotes the size of the balance sheet of the biggest bank in the system,  $\alpha$ ,  $\beta$  and  $z$  are constants, and  $\theta$  is the classic Heaviside step function. Section 3 will present the main topological properties of networks produced by functions (7), (8) and (9). With any of these probability functions, we can build the  $n \times n$  probability matrix  $P \in M_{n \times n}$ , with entries  $p_{ij} = P_s(A_i, A_j) \in [0, 1]$ , and  $s = 1, 2, 3$ <sup>1</sup>;

4. the next step consists in constructing the adjacency matrix  $A$  of the network, according to the rule:

$$a_{ij} = \begin{cases} 1, & \text{with probability } p_{ij} \\ 0, & \text{with probability } (1 - p_{ij}) \end{cases}$$

---

<sup>1</sup> We note here that random networks are a particular case of our generating algorithm; in fact, each function with the form  $P(A_i, A_j) = p$  will generate random networks characterized by a density  $p$ .

In contrast to a standard random network with constant connectivity, the probabilities  $p_{ij}$  are drawn from one of the probability functions of eqs. (7) to (9). In this way we reproduce the systematic tendency of accumulation of links at larger entities and the disassortative nature of empirical banking networks <sup>2</sup>;

5. we also assume that banks loan money to other banks according to their peculiarity; since loans are supposed to produce returns, it seems natural to assume that financial entities will have more intense links with banks with high peculiarity (balance sheet size). Including this notion in the probability functions we can compute the load  $l_{ij}$  (the volume of credit) on the link between bank  $i$  and bank  $j$  as:

$$l_{ij} = \frac{l_i p_{ij}}{\sum_{j \in \Omega_i} p_{ij}} \quad (10)$$

where  $\Omega_i$  denotes the set of nodes for which  $a_{ij} = 1$ ;

6. in the last step, we compute the internal borrowing  $b_i$  as:

$$b_i = \sum_j l_{ji} \quad (11)$$

and the customers' deposits  $d_i$  as:

$$d_i = (e_i + l_i) - (\eta_i + b_i) \quad (12)$$

Deposits are, thus, the residual in the construction of the balance sheets of banks that is adjusted in a way to guarantee consistency. While this leads to a certain degree of heterogeneity of the size of deposits across banks, this is not necessarily an unrealistic feature of our system.

Let us also emphasize that in the algorithm there are two levels of randomness: the first appears in step 1, in the determination of the sizes of the nodes, while the second appears in step 4, in the realization of the probability matrix. Thus, for a fixed sequence of the sizes  $\{A_i\}$ , several different realizations of the network are possible.

### 3 Topological properties and the probability function

The representation of the financial system in our model depends on the choice of the probability function. In the following, we will show in detail

---

<sup>2</sup> It is possible, especially for symmetric probability functions, to have situations where  $a_{ij} = a_{ji} = 1$ . Since loops are not allowed in our model (they would mean that bank  $i$  and  $j$  are both borrower and lender of each others), we have to use a criterion for the elimination of one of the edges. A possible choice is to randomly eliminate one of the two links  $i \rightarrow j$  or  $j \rightarrow i$ ; however, other choices are possible as well, if the aim is to enforce the disassortative behavior of the networks (see section 3).

how the topological structure of the network is determined by functions (7), (8) and (9). One of the main features of these kind of networks is the presence of power laws in the degree distributions of both *in-* and *out-degree*. In particular, it is easy to see that the relations between the probability function and the degree distributions are <sup>3</sup>:

$$P(k_{in}) = \rho \left[ F_{in}^{-1} \left( \frac{k_{in}}{n} \right) \right] \cdot \frac{d}{dk_{in}} F_{in}^{-1} \left( \frac{k_{in}}{n} \right) \quad (13)$$

$$P(k_{out}) = \rho \left[ F_{out}^{-1} \left( \frac{k_{out}}{n} \right) \right] \cdot \frac{d}{dk_{out}} F_{out}^{-1} \left( \frac{k_{out}}{n} \right) \quad (14)$$

where  $n$  is the number of nodes in the network,

$$n \cdot F_{in}(A_i) = k_{in}(A_i) = n \cdot \int_a^b P_s(t, A_i) \rho(t) dt \quad (15)$$

is the mean *in - degree* depending on the fitness parameter  $A_i$  and

$$n \cdot F_{out}(A_i) = k_{out}(A_i) = n \cdot \int_a^b P_s(A_i, t) \rho(t) dt \quad (16)$$

is the mean *out - degree*. In the above equations,  $a$  and  $b$  denote respectively the lower and the upper limits for the support of the distribution of bank sizes:  $A_i \in [a, b]$ . With probability functions 7, 8 and 9, we obtain respectively:

$$P_1(k_{in}) \propto k_{in}^{-\frac{1+\beta}{\beta}}, \quad P_1(k_{out}) \propto k_{out}^{-\frac{1+\alpha}{\alpha}} \quad (17)$$

$$P_2(k_{in}) \propto (c_1 k_{in} + c_2)^{-2}, \quad P_2(k_{out}) \propto (c_3 k_{out} + c_4)^{-2} \quad (18)$$

$$P_3(k_{in}) \propto k_{in}^{-2}, \quad P_3(k_{out}) \propto k_{out}^{-2} \quad (19)$$

In the same way, it is possible to see that the average degree of a neighbour is determined by:

$$\langle k_{nn} \rangle (A_i) = \frac{N}{k(A_i)} \cdot \int_a^b p(A_i, t) k(t) \rho(t) dt \quad (20)$$

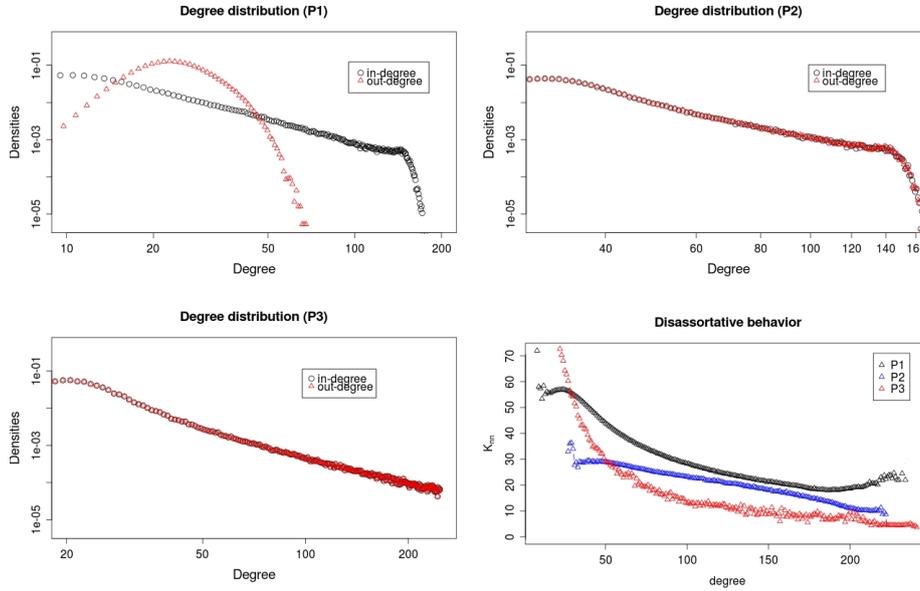
where  $k(A_i)$  is the mean *total degree* of node  $i$ , as a function of its own fitness parameter.

As we can see, with all three kinds of probability functions, the results are scale-free networks (i.e., a power-law distribution of degrees). Since eq.

---

<sup>3</sup> The derivation of the following equations, well known in literature (see for example Caldarelli, 2007), is also reported in the Appendix.

(20) involves the mean total-degree of a node,  $k(A_i)$ , there is no closed-form solution for this expression for the three probability functions. The disassortative behavior can, however, be confirmed via numerical integration of eq. (20), cf Fig. 2. It is apparent from eq. (17) to (19), that it will be possible to change the exact shape of the degree distributions as well as the degree of disassortative behavior by modifying the parameters of the probability functions, and the distribution of the fitness parameters. Figure 2 shows the degree distributions and the average neighbour degree for functions (7), (8) and (9), for parameters  $\alpha = 0.25$ ,  $\beta = 1$  and  $z = 0.6 \cdot A_{max}$ . With this choice of the parameters we get tail indices in the in-degree distribution equal to, respectively,  $-2$ ,  $-2$  and  $-2$ , and  $-5$ ,  $-2$  and  $-2$  for the out-degree distributions. Moreover, a clear disassortative behavior is observed in all the three cases<sup>4</sup>.



**Fig. 2:** The first three panels show the *in-* and *out-degree* distributions for the three probability functions (7), (8) and (9). The last panel shows the mean neighbour degree as a function of the *total degree* of the nodes. The curves in the last panel are decreasing with the degree itself, indicating that big nodes are connected to a multitude of small and medium-sized nodes, which themselves are connected with only a (relatively) small number of hubs.

<sup>4</sup> In order to reinforce the disassortative behavior, one could use a criterion for the elimination of the loops different from the one described in footnote 2. In particular, if both the edges  $i \rightarrow j$  and  $j \rightarrow i$  are present in the network, one could eliminate the one starting from the biggest node of the two: this mechanism would contribute to mimicking real interbank network structures, where mostly small banks lend money to big banks, as described in the introduction.

## 4 Simulation results

In this section we present results from our simulation engine. The design of the simulations will be the same for all the following experiments: the first step consists in generating a Monte Carlo realization of our banking system as explained in sec. 2. In the second step we destroy the largest bank: this shock is assumed to wipe out all the external assets from the balance sheet of the initially failing bank. For each simulation run, we count the overall number of defaults, as well as the number of defaults in each single phase of the shock propagation. We report the average number of defaults across all banks. In the following the number of banks will be fixed at 250, and we will use probability functions (7), (8) and (9) with parameters  $\alpha = 0.25$ ,  $\beta = 1$  and  $z = 0.6 \cdot A_{max}$ ; furthermore the two limits  $a$  and  $b$  will be fixed at 5 and 100 respectively. We will investigate later how those limits affect the resilience of the system. Of course, other choices are possible both for the parameters and for the probability functions.

In the following, we will initially use as our exemplary case probability function (7): for systems produced by equation (7) we will vary only one parameter at a time, and we will study how this changes the domino effects. At the end of the section we will show for all the functions (7), (8) and (9) the results obtained by varying simultaneously the percentage of net worth  $\eta$  and the percentage of interbank borrowing  $\theta$ . Moreover, in section 4.4 we present a comparison between the result obtained with scale free networks and the results obtained with random networks and networks generated via a maximum entropy principle. Section 4.5 shows how the absolute size of the largest hub affects the contagion process.

### 4.1 Transmission of shock

Here we study the consequences of an idiosyncratic shock hitting one of the banks in the system, and elaborate on how the aftereffects (usually the number of defaults) depend on the structural parameters of the system. There are several ways in which a shock can propagate through a financial system. First, propagation will occur through the direct bilateral exposure between banks (namely, financial entities holding in their balance sheets liabilities of other entities and incurring, for endogenous reasons, solvency problems, will transmit their losses to their creditors), correlated exposure of banks to a common source of risk (banks holding correlated portfolios can increase the probability of multiple and simultaneous failures), effects arising from endogenous fire-sales of assets by entities in distress, and informational contagion. We will focus here on the first of those mechanisms, noting that idiosyncratic shocks are a clear starting point for studying knock-on defaults due to interbank exposure.

In our subsequent analysis, the shock starts from one bank, and it con-

sists in wiping out a certain percentage of its external assets (the *source* of the shock). Let  $p_i$  be that percentage, and let  $s_i$  be the size of the initial shock:

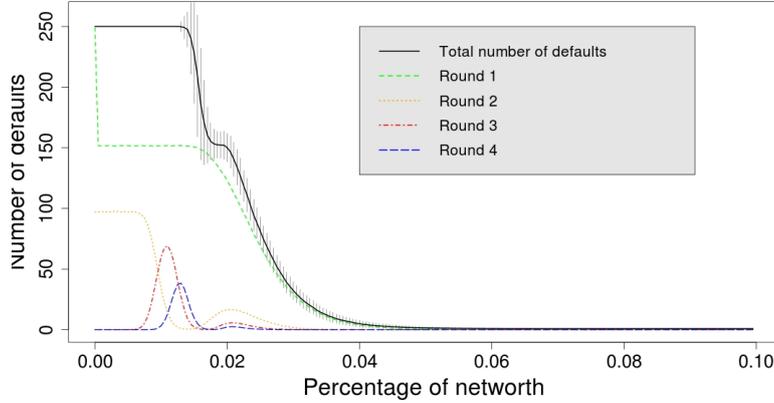
$$s_i = p_i \cdot e_i \quad (21)$$

This loss is first absorbed by the bank's net worth  $\eta_i$ , then its interbank liabilities  $b_i$  and last its deposits  $d_i$ , as the ultimate *sink*. That is, we assume priority of (insured) customer deposits over bank deposits which, in turn, take priority over equity (net worth). If the bank's net worth is not big enough to absorb the initial shock, the bank defaults and the residual is transmitted to creditor banks through interbank liabilities. And in case these liabilities are not large enough to absorb the shock, some of the losses have to be absorbed by depositors. Formally, if  $s_i > \gamma_i$ , then bank  $i$  defaults. If the residual loss  $(s_i - \gamma_i)$  is less than the interbank borrowing  $b_i$  of the failed bank, then all residual loss is transmitted to creditor banks. Otherwise, if  $(s_i - \gamma_i) > b_i$ , then all of the residual cannot be transmitted to creditor banks and depositors receive a loss of  $(s_i - \gamma_i - b_i)$ . Creditor banks receive an amount of the residual shock proportional to their exposure to the failed bank. In turn, this loss is first absorbed by their net worth. If their net worth is not big enough to completely absorb the shock, it will be transmitted first to their creditors bank, and possibly also to their depositors. The part that is transmitted through the interbank channels may cause further rounds of contagious defaults, and in this way the shock spreads through the network. The transmission continues spreading through the system until the shock is completely absorbed or, alternatively, the system has completely failed. In the following, we will consider always the worst situation, namely that all the external assets of one bank are wiped out:  $p_i = 1$ . For our analysis of the mechanical short-run effects of a shock this is not an unrealistic assumption. Partial recovery of claims to defaulted entities requires certain legal proceedings that can be extremely time consuming. Over short horizons, the *de facto* situation is that no payment can be enforced on a defaulted claim.

## 4.2 Bank capitalization

In this first experiment we investigate the effects of banks' net worth on the resilience of the entire banking system; the parameter  $\theta$  will be fixed at 0.8, so that each bank will invest 20% of its total assets in the interbank market, and the remaining 80% in some external markets. We will let the parameter  $\eta$  vary from 0 to 0.1<sup>5</sup>. Figure 3 shows the result: we report both the total number of defaults (black bold line), and the number of defaults in the first four phases of the propagation of the shock. The thin vertical bars represent

<sup>5</sup> Remember that by mere rescaling  $\eta$  could also be interpreted as the excess over the required minimal capital requirement.



**Fig. 3:** Number of defaults as a function of the percentage of net worth  $\eta$ , for probability function (7). The other parameters are fixed at:  $\theta = 0.8$ ,  $a = 5$ ,  $b = 100$ . The picture shows both the total number of defaults (bold black line) together with the standard deviation of the mean value (thin gray line), and the number of defaults occurring during the first four phases of the propagation of the shock.

the standard deviation of the black line across our 200 replications of the simulations.

As one could expect, when the percentage of net worth tends to zero, the total number of defaults increases to 250: in particular, a threshold value ( $\eta = 0.0143$  in the picture) exists below which the system fails completely, and below  $\eta = 0.008$  it breaks down within only two rounds. This is a demonstration on the so called *small-world* effect: the diameter of this kind of networks is roughly about two when measured from the largest bank belonging to the system, and so in only two rounds the shock will have reached almost any bank of the IBM. At the other end, when the percentage of net worth is beyond an upper threshold value, no defaults are reported and no domino effects set in.

Interestingly, the shape of the line describing the total number of defaults is far from linear. Starting from the value  $\eta = 0.1$ , we can observe that below the value  $\eta \cong 0.05$  the first defaults appear, and inspection shows that these are typically small banks connected to the initially failing bank. As  $\eta$  decreases further, we observe a sharp increase in the number of defaults, and this growth stops at the value  $\eta \cong 0.02$  where the curve enters a *plateau*: at this point, all the banks belonging to the first *shell* around the initially failing bank have failed, and the banks which are not directly connected to the first failing unit have enough net worth to survive the shock. As the net worth decreases further, also the banks outside the first shell are no more able to absorb the perturbation, and the total number of defaults sharply

moves up to 250.

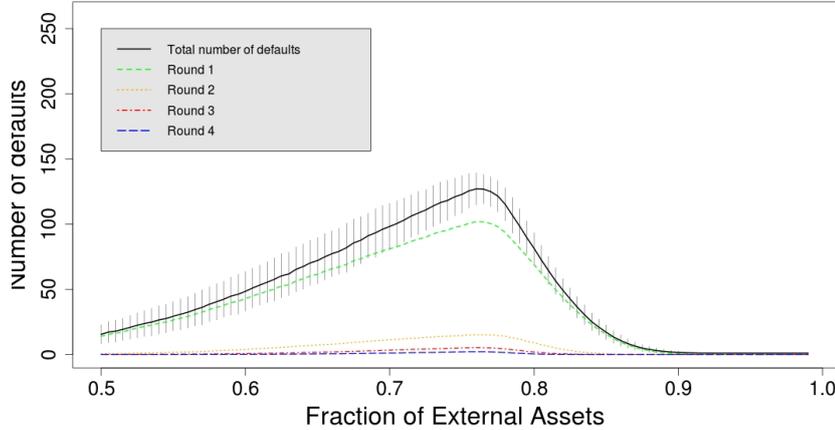
It is interesting to have a look at the number of defaults in the different rounds. In the first round (dotted line in Figure 3), banks that fail are directly connected to the initially shocked bank, and when the dotted line reaches its *saturation* at  $\eta \cong 0.018$  the complete first *shell* (composed on average of 153 units) has failed. We note that the saturation point of the number of defaults in the first round does not coincide exactly with the *plateau* of the total number of defaults: the explanation is that the largest banks in the first shell need more than one hit to fail, and so they populate the failures of higher rounds. The reason for this is that for larger banks the overall number of credit relationships to other banks is larger too (by assumption, following observed empirical regularities), and so for them the failure of the largest bank will lead to a proportionally smaller loss than for the smaller client banks of the defaulted entity. When the percentage of net worth decreases, these defaults occur already in earlier rounds, up to a point in which all banks of the first shell are affected in the first round of defaults.

It is worthwhile to highlight here the ability of the system to confine the shock in the first shell if the value of  $\eta$  is higher than some benchmark (approximately 0.018 in our example). Even if contagion defaults occur after the first failure these are limited to banks inside the first shell, i.e. to those banks with direct exposure towards the source of the disruption.

### 4.3 Interbank exposure

In this section we are going to explore how the number of defaults is affected by the percentage of interbank exposure as a function of total assets, namely how the parameter  $\theta$  affects the resilience of the system. An increase in interbank assets produces, as an immediate result, an increase in the weight of each edge, and so an increase of the channels through which the shock can propagate. This effect can potentially increase the number of defaults in the system, as the amount of losses transmitted to creditor banks will increase as well. On the other hand, an increase in interbank exposure implies a reduced relative exposure to external markets, and since here we are considering, as initial source of the shock, the external assets, this second effects could cushion banks against systemic risk.

The design of the simulations will remain the same as in the first experiment: we generate a realization of the system and we shock the biggest bank, wiping out all its external assets. Subsequently we count the number of defaults. We will show the mean value of those numbers for each round, and the standard deviation for the total number of defaults. In this section, the percentage of net worth  $\eta$  is fixed at 0.025, while the percentage of external assets on total assets,  $\theta$ , varies from 0.5 to 1 (when  $\theta$  is equal to one no interbank assets are present in the bank balance sheets). Fig. 4 shows the result.



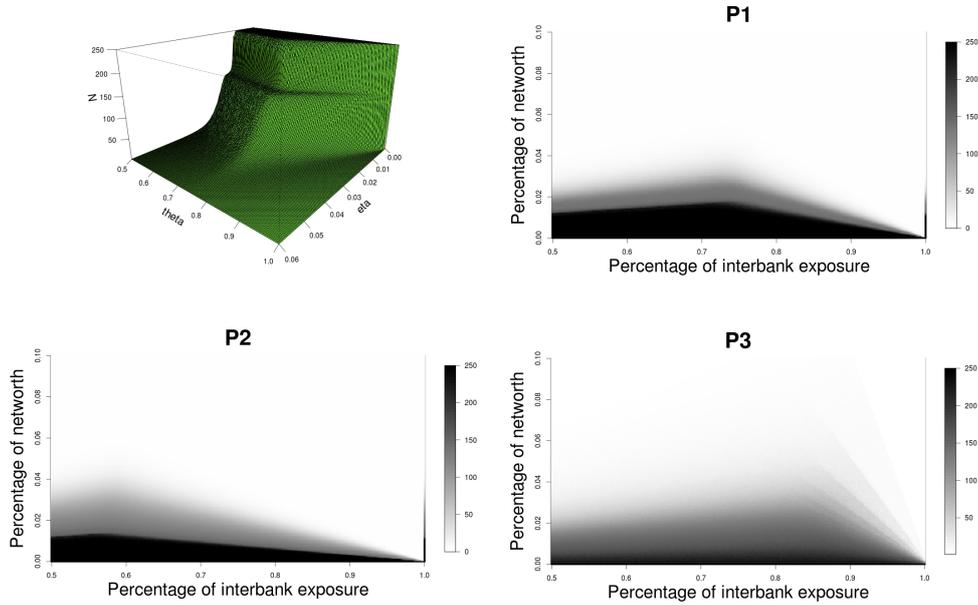
**Fig. 4:** Number of defaults as a function of the percentage of external assets  $\theta$ , i.e.  $1 -$  the percentage of interbank exposure  $\theta$ , for probability function (7). The other parameters are fixed at:  $\eta = 0.025$ ,  $a = 5$ ,  $b = 100$ . The picture shows both the total number of defaults (bold black line) together with the standard deviation of the mean value (tiny grey line), and the number of defaults occurring during the first four phases of the propagation of the shock.

First, we note that when  $\theta$  tends to 1 the number of defaults tends to zero: in this case the banks' balance sheets contain only external assets, and so the channels for the propagation of the shock become smaller and smaller, until  $\theta$  assumes the value 1 and there are no more links in the network, and no domino effects are possible. In Fig. 4 we can also note a threshold value at  $\theta \cong 0.78$ : at this value, the contagion effects reach their maximum while both more or less intense interbank linkages reduce the number of knock-on defaults (due to a higher degree of risk sharing on the left and fewer links for contagion on the right). At the other extreme, when  $\theta$  tends to 0, the banks become completely isolated from any external market, and so in our model, where the initial source of the shock comes from the external assets of the largest bank of the system, the number of defaults tends to zero as well. Note that this exercise does not leave the size of the internal shock unaffected. Clearly, when external assets decline in their absolute size (from right to left) there should be a decrease of contagious defaults. Nevertheless, despite this lack of normalization of the shock, the behavior of the system is distinctly non-monotonic.

#### 4.4 Results with other network generators

So far we have always used eq. (7) as the probability function generating the networks. Although eq. (7) correctly reproduces the disassortative behavior

and power law degree distributions, it is interesting to see in how far other functional forms generating systems with the same qualitative features reproduce the above results or not. We are going to present, therefore, also the results obtained for the other two functions, namely equations (8) and (9). In this section we display results in the bidimensional space  $(\eta, \theta)$ , and for each pair of these two parameters we use colors to indicate the number of defaults. Figure 5 shows the results for the three probability functions discussed in section 3.



**Fig. 5:** In the top left, a 3D plot shows the total number of defaults as a function of the two parameters  $\eta$  and  $\theta$  for probability function  $P1$ : in the figure one again detects the *plateau* that already appeared in Fig. 3. The other three colored maps represent the same information for the three probability functions  $P_1$ ,  $P_2$  and  $P_3$ . In all these maps one observes a non-monotonic behavior of defaults in the percentage of interbank exposure. The color code indicates the number of banks in default.

As one can see from the Figure, the behavior of the systems in the presence of a perturbation is qualitatively the same in all the three cases. In particular, it is again possible to observe a threshold value for the percentage of interbank exposure  $\theta$ , beyond which the trend in the total number of defaults reverts itself. Different versions of our generating mechanisms for interbank connections do, however, affect the location of the level of interbank exposure leading to the largest level of fragility of the system as well as the quantitative importance of defaults in higher rounds.

As we had already highlighted in the introduction of this paper, most empirical and simulation-based approaches of interbank markets use as topol-

ogy for the underlying bank network a random network or a maximum entropy principle. Random networks are characterized by a constant probability  $p$  for each edge to exist in the network. The maximum entropy principle, on the other hand, assumes a maximum of dispersion of inter-bank loans (see Upper and Worms (2004) for more details on this kind of networks). We want to compare here the differences in term of contagion effects when the same set of banks is connected through different underlying network structures.

In the following, we will compare the number of defaults in scale free networks, random networks and networks generated via maximum entropy principles, for varying capitalization of the system. For the scale free network case, we use as benchmark case the system generated through function (7): again, the limits  $(a, b)$  are set to  $(5, 100)$  and the parameter  $\theta$  is fixed to 0.8.

For the random network case, we will simply use the probability function:

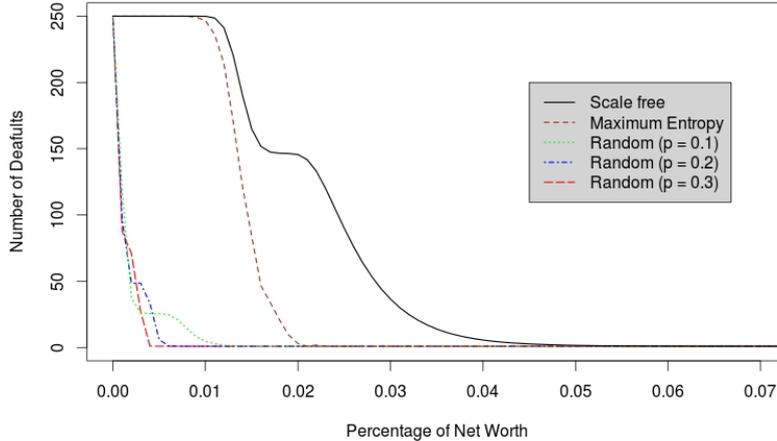
$$P(A_i, A_j) = \begin{cases} p, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases} \quad (22)$$

with  $p$  equal to 0.1, 0.2 and 0.3<sup>6</sup> We note here that with the value  $p = 0.1$ , the (mean) number of edges in the system generated with function (22) is equal to the (mean) number of edges in system generated with function (7): this is equivalent to random reshuffling the links (and their weights) among all the banks.

We cannot define a probability function that generates networks according to the maximum entropy principle. For a consistent comparison with the scale free scenario, we proceed in the following way: first we generate a weight matrix  $W$  using the fitness algorithm described in section 2 (with probability function given by eq. (7)), then we compute the sum of the rows and the sum of the columns of that matrix: they are, respectively, the total amount of interbank borrowing and the total amount of interbank lending for each bank. The problem is then to determine a new weight matrix  $W^*$  such that (i) the sum of the rows and the columns are the same as for  $W$ , and (ii) the dispersion of the new bilateral exposures  $w_{ij}^*$  is maximized. This problem can be easily solved numerically using the RAS algorithm (see Censor and Zenos (1997) for technical details). The result is a banking system populated by banks having exactly the same balance sheets as in the scale free network case, but now connected in a way that maximizes the entropy of the new weighted matrix  $W^*$ .

Figure 6 shows the results. As in the previous simulations, we again shock the largest bank in the system by wiping out all its external assets. The figure shows the total number of defaults after the propagation of the shock terminates (for better visibility, we do not report in this graph the

<sup>6</sup> This will simply generate random networks with different densities.



**Fig. 6:** Number of defaults as a function of the percentage of net worth for different kinds of network topologies: scale-free networks, networks designed according to the maximum entropy scenario and three random network scenarios with different probability for the existence of links (the random networks generated with  $p = 0.1$  have the same (mean) density as in the scale free case).

standard deviations). We note immediately from the figure that the scale free scenario is the most critical in terms of number of defaults. The random network scenario (no matter what the probability  $p$  is) always underestimates the effect of a targeted attack: the large pool of small and medium-sized banks now has a larger number of outgoing links randomly directed to all the other banks in the system, and, for each bank, the weight on those links is the same (in contrast to the scale free scenario, where the larger the peculiarity of the node, the larger the weight on the links pointing to it). This effect dramatically reduces the threshold value for the percentage of net worth necessary for triggering chains of defaults.

We note moreover that also the maximum entropy scenario underestimates the effects of a targeted attack, albeit to a smaller extent in comparison to the random networks. We see that the classical *plateau* that we have seen in all the other cases now disappears: the reason is that the systems built via the maximum entropy principle are fully connected<sup>7</sup>, and so the distinction between different shells is not applicable here, i.e. all banks belong to the first shell.

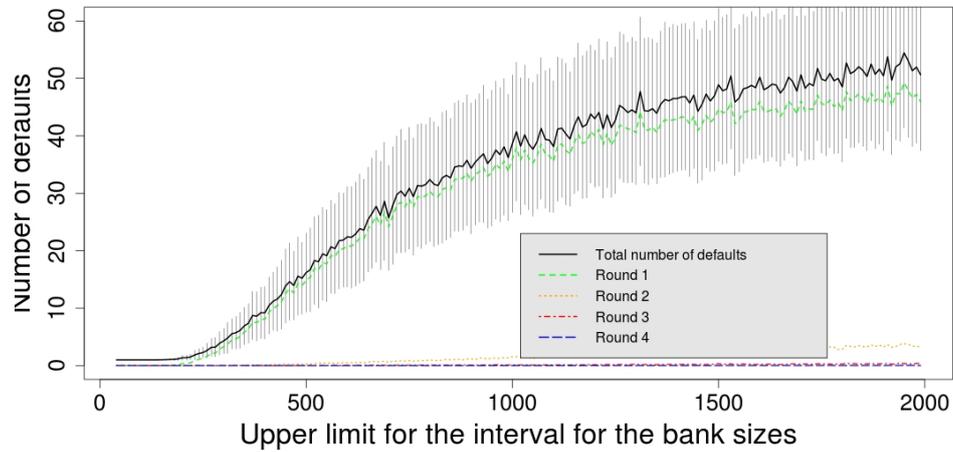
<sup>7</sup> Note that the result is not equivalent to use a random network with probability  $p = 1$ , since the weights on the links are significantly different, affecting so the way a shock can propagate in the system.

## 4.5 The size of the hubs

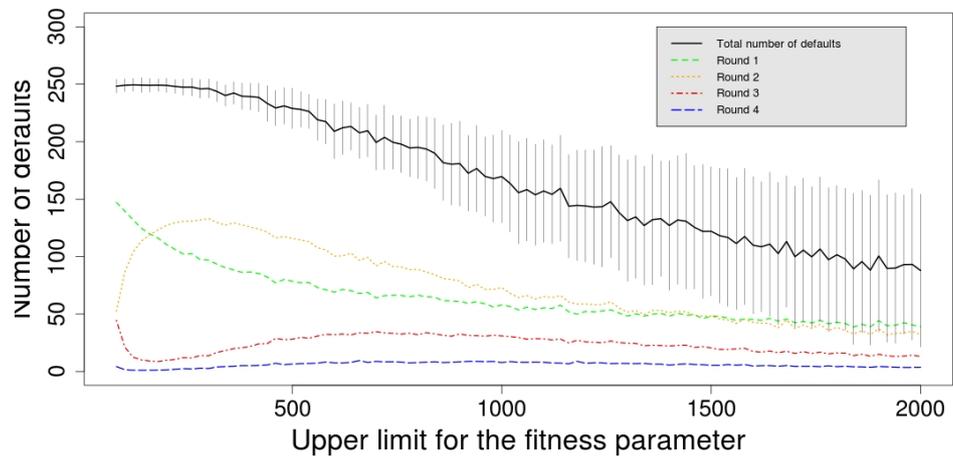
In this section, we analyze the behavior of the networks when changing the size of the largest bank in the system. Using our simulation engine, this can be achieved simply by expanding the interval from which we draw the fitness parameters of nodes. In particular, we leave the lower boundary  $a$  of that interval constant (in our experiments it will be (and it was) fixed to 5), and increase the upper limit  $b$ . Since the fitness parameters are drawn from a power law distribution most of the sampled values will be located in a short subset at the left end of the interval. As an example, we can imagine to sample the fitness parameters from a power law on the interval  $[5, 1000]$ . If the exponent is 2 it is easy to see that more than 95% of the draws will lie in the interval  $[5, 100]$ , and that 99.5% of the values will lie in the interval  $[5, 500]$ . So the result of increasing the upper limit of the interval  $[a, b]$  is the introduction of a very small number of very big banks. The presence of those banks has some intuitively plausible effects on the resistance of the system to shocks. Consider, for example, the probability function of eq. (7), with  $\alpha = 0.25$  and  $\beta = 1$ . When  $A_{max}$  increases, the probability of a link involving two small banks or two medium-sized banks decreases: hence, more edges will point to the few hubs of the networks. Furthermore, since the edges in our model are weighted by the same probability function (see eq. 10), most of the interbank loans will be loaded on the edges pointing to these hubs.

After these preliminary considerations, we now investigate the behavior of the network when bigger hubs are introduced in the interbank system. We will show results for two particular values for the percentage of (excess) net worth  $\eta$ , namely  $\eta = 0.1$  and  $\eta = 0.01$ . These choices permit us to study the system in two limiting cases: in the first case, as demonstrated in the previous sections, the system is relatively well cushioned against systemic risk, while in the second case the system is very *weak*. The parameter  $\theta$  will be fixed at the value 0.8. Furthermore, for each realization of the system, we will again shock the largest bank by wiping out all its external assets from its balance sheet.

Fig 7 shows the results for the case  $\eta = 0.1$ . As a first observation, we note that as the value of the upper limit  $b$  exceeds the threshold value  $b \cong 230$ , first round effects start. We should emphasize that, in the previous experiments, at  $(\eta, \theta) = (0.1, 0.8)$  no defaults were reported in our system. Figure 7 shows that occurrence or not of contagious defaults also depends on the parameter  $b$ . In particular, we can see that the number of defaults in the first round sharply increases in the range  $[230, 700]$ . That happens because most of the banks are now linked to the hubs and moreover these links become increasingly more loaded (higher in volume) as the parameter  $b$  increases. As a consequence, when the largest bank defaults, the first shell is no longer able to absorb the resulting losses.



**Fig. 7:** Number of defaults as function of the upper limit of the interval for banks' sizes used in the Monte Carlo simulations. The black bold line denotes the total number of defaults, together with its standard deviation (light gray bars); colored lines represent the next rounds. ( $\eta = 0.1$ ,  $\theta = 0.8$ ).



**Fig. 8:** Number of defaults as function of the upper limit of the interval for banks' sizes used in the Monte Carlo simulations. The black bold line denotes the total number of defaults, together with its standard deviation (light gray bars); colored lines represent subsequent rounds. ( $\eta = 0.01$ ,  $\theta = 0.8$ ).

Figure 8 shows the result in the case  $\eta = 0.01$ ,  $\theta = 0.8$ . For this pair of parameters we know that the system is extremely vulnerable against systemic risk, and in particular in a few rounds the whole IbM usually had failed after a shock. As one can see from Figure 8, these results change as well if larger hubs are present in the system: as the size of the biggest bank becomes larger, the pool of small and medium-sized banks effectively stops dealing among themselves, and so the channels through which a shock can propagate beyond the (relatively large) pool of trading partners of the largest hub do vanish. We can see moreover that as the upper limit  $b$  increases, the second round (yellow line in Figure 8) assumes basically the same importance as the first round. At this point in the system there are few big hubs (strongly) interconnected, and when the biggest of them fails (producing the default of all banks in its first shell) the other hubs will be failing in due course. When these secondary hubs fail, their first shells will fail as well, and the result is a high number of defaults in the second round. As the upper limit  $b$  increases further, networks will become very sparse, and the number of defaults decreases due to lower overall connectivity of the system.

## 5 Conclusion

This paper has investigated the behavior of a scale-free interbank market, characterized by a disassortative structure, in case of targeted attacks. The networks have been constructed according to a *fitness* algorithm, where the size of each node is used as a kind of peculiarity index for the bank itself: the higher the index, the higher the probability that other banks will lend money to it. For appropriate choices of the probability function, the networks are described by a decreasing mean neighbour degree distribution, i.e. disassortative mixing. The results are networks composed of a large pool of small and medium-sized banks which invest money in interbank loans to the biggest banks, which in turn invest this liquidity into non-financial assets and also redistribute part of it in the interbank market.

In this framework, we have investigated how the percentage of net worth and the percentage of interbank assets (on total assets) affects the spread of an idiosyncratic shock. The results show a shell structure in the propagation of losses: banks belonging to the first shell (i.e. creditor banks of the defaulted entity) fail mostly before the others, and it is possible to distinguish between defaults of the different shells in the cascade of events. Moreover, in all three types of probability functions we investigated, a hump-shaped dependency of the number of defaults on  $\theta$  was observed, indicating higher robustness of networks with very few and very many links. The intuitive explanation is that if banks invest more money in the interbank market than in other external markets, the risk for endogenous shocks decreases and, moreover, banks are more able to absorb potential losses.

As it turns out, the role of the hubs is ambiguous in these networks: when

the size of the hubs increases, the pool of small and medium-sized banks tends to withdraw from dealing among themselves, and to start lending and borrowing mostly from and to the hubs. Given our probability functions, the hubs are also highly connected among themselves. The results of this change in the network structure on the resilience of the system is linked to two antagonistic phenomena: on one hand, the number of channels for the shock propagation decreases as the hub sizes increase. Due to the smaller number of connections in the pool of small and medium sized banks, on the other hand, the same pool of banks concentrates their bilateral links to a few very big banks, that assume a central position in the entire system. In our model, the results from an endogenous shock are ambiguous and depend on the state of the system in terms of its capital base: for a *strong* system ( $\gamma = 0.1$ ) the total number of defaults increases if the biggest bank meets insolvency problems, for *weakly* capitalized system ( $\gamma = 0.01$ ), the number of defaults decreases with the size of the largest unit.

We also found that random networks or networks constructed on the base of a maximum entropy principle lead to fewer contagious defaults than our scale-free networks, under otherwise identical conditions. It is important to note that this implies a potentially tremendous underestimation of contagion risk, if due to a lack of detailed knowledge, stress tests are conducted with the simple algorithms for random network creation or maximum entropy allocation of interbank credit.

## A Computation of the degree distribution via the probability function

We provide here the derivation of eq.s 13 and 14. Starting from a particular probability function  $P_S(A_i, A_j)$ , and a distribution for the *size* parameter  $\rho(A_i)$ , we can write the mean in-degree of a vertex as:

$$k_{in}(A_i) = n \int_a^b P_S(t, A_i) \rho(t) dt = n \cdot F_{in}(A_i) \quad (23)$$

and, similarly, for the out-degree we can write:

$$k_{out}(A_i) = n \int_a^b P_S(A_i, t) \rho(t) dt = n \cdot F_{out}(A_i) \quad (24)$$

where  $n$  is the number of nodes of the network. Assuming the function  $F_{in}(A_i)$  and  $F_{out}(A_i)$  to be monotonous in  $A_i$ , and for  $n$  large enough, we can invert the functions  $F_{in}$  and  $F_{out}$  in order to find the relationships between the size parameter  $A_i$  and the the out-and in-degree of the node:

$$A_i = F_{in}^{-1} \left( \frac{k_{in}}{n} \right) \quad (25)$$

$$A_i = F_{out}^{-1} \left( \frac{k_{out}}{n} \right) \quad (26)$$

The transformation of the parameter in the size-distribution  $\rho(A_i)$ , from  $A_i$  to  $k_{in/out}$ , bring us to:

$$P(k_{in}) = \rho \left[ F_{in}^{-1} \left( \frac{k_{in}}{n} \right) \right] \cdot \frac{d}{dk_{in}} F_{in}^{-1} \left( \frac{k_{in}}{n} \right) \quad (27)$$

$$P(k_{out}) = \rho \left[ F_{out}^{-1} \left( \frac{k_{out}}{n} \right) \right] \cdot \frac{d}{dk_{out}} F_{out}^{-1} \left( \frac{k_{out}}{n} \right) \quad (28)$$

## References

- [1] Allen F, Gale D (2000) *Financial contagion*. Journal of Political Economy, Vol. 108, pages 1-33.
- [2] Battiston S, Delli Gatti D, Gallegati M, Greenwald B, Stiglitz J E (2009) *Liaisons dangereuses: increasing connectivity, risk sharing, and systemic risk..* Working Paper.
- [3] Benito E (2008) *Size, Growth and Bank Dynamics*. Documento de Trabajo, Banco De España.
- [4] Boss M, Elsinger H, Summer M, Thurner S (2004) *Network topology of the interbank market*. Quantitative Finance, 4(6), 677-684.
- [5] Caladrelli G, (2007) *Scale-Free Networks. Complex web in nature and technology*. Oxford University Press.
- [6] Censor Y, Zenios S A (1997) *Parallel optimization, Oxford University Press*.
- [7] Cocco J F, Gomes F J, Martins N C (2009) *Lending relationships in the interbank market*. Journal of Financial Intermediation, 18 2448.
- [8] De Masi G, Iori G, Caldarelli G (2006) *Fitness model for the Italian interbank money market*. Physical Review E 74, 066112.
- [9] Ennis H M (2001) *On the size distribution of banks*. Federal Reserve Bank of Richmond Economic Quarterly, 87, pp. 1-25.
- [10] Finger K, Fricke D, Lux T (2012) *Network Analysis of the e-MID Overnight Money Market: The Informational Value of Different Aggregation Levels for Intrinsic Dynamic Processes*. Kiel Working Paper, 1782, Kiel Institute for the World Economy, Kiel.

- [11] Gai P, Kapadia S (2010) *Contagion in Financial Networks*. Working paper no 383/2010, Bank of England.
- [12] Gai P, Haldane A, Kapadia S (2011) *Complexity, concentration and contagion*. Journal of Monetary Economics, Volume 58, Issue 5, Pages 453–470.
- [13] Haldane A, May R (2011) *Financial Systems: Ecology and Economics*. Nature 469.
- [14] Imakubo K, Soejima Y (2006) *The Transaction Network in Japan's Interbank Money Market*. Financial market department, Bank of Japan, *Financial System Report*.
- [15] Inaoka H, Ninomiya T, Taniguchi K, Shimizu T, Takayasu H (2004) *Fractal Network Derived from Banking Transaction: An Analysis of Network Structures Formed by Financial Institutions*. Bank of Japan Working Paper No. 04-E-4, Bank of Japan.
- [16] Iori G, de Masi G, Precup O, Gabbi G, Caldarelli G (2008) *A Network Analysis of the Italian Overnight Money Market*. Journal of Economics and Dynamics & Control, 32, pp. 259–278.
- [17] Janicki H P, Prescott E C (2006) *Changes in the size distribution of U.S. banks: 1960-2005*. Federal Reserve Bank of Richmond Economic Quarterly, 92, pp. 291-316.
- [18] Lux T, Fricke D (2012) *Core-Periphery Structure in the Overnight Money Market: Evidence from the e-MID Trading Platform*. Kiel Working Paper, 1759, Kiel Institute for the World Economy, Kiel.
- [19] May R, Arinaminpathy N (2010) *Systemic Risk: The Dynamics of Model Banking Systems*. Journal of the Royal Society Interface 2.
- [20] Memmel C, Sachs A (2011) *Contagion in the interbank market and its determinants*. Deutsche Bundesbank, *Discussion Paper, Series 2: Banking and Financial Studies*. No 17/2011.
- [21] Mistrulli P E (2007) *Assessing financial contagion in the interbank market: Maximum entropy versus observed interbank lending patterns*. Temi di discussione, N 641 (working papers).
- [22] Nier E, Yang J, Yorulmazer T, Alentorn A (2007) *Network models and financial stability*. Journal of Economic Dynamics & Control 31.

- [23] Soramäki K, Bech M L, Arnold J, Glass R J, Beyeler W (2006) *The topology of Interbank Payment Flows*. Staff Report, 243, Federal Reserve Bank of New York.
- [24] Upper C (2011) *Simulation methods to assess the danger of contagion in interbank markets*. Journal of Financial Stability, doi:10.1016/j.jfs.2010.12.001.
- [25] Upper C, Worms A (2004) *Estimating Bilateral Exposures in the German Interbank Market: Is there a Danger of Contagion?* European Economic Review, 8, 827-49 09/02.