Additive Utility in Prospect Theory

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Abstract. Prospect theory is currently the main descriptive theory of decision under uncertainty. It generalizes expected utility by introducing nonlinear decision weighting and loss aversion. A difficulty in the study of multiattribute utility under prospect theory is to determine when an attribute yields a gain or a loss. One possibility, which has been adopted in the existing theoretical literature on multiattribute utility under prospect theory, is to assume that a decision maker views the complete outcome as a gain or a loss. In this *holistic approach*, decision weighting and loss aversion are general and attribute-independent. Another possibility, more common in the empirical literature, is to assume that a decision maker has a reference point for each attribute. We give preference foundations for this *segregated approach* in which decision weighting and loss aversion are attribute-specific.

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1 Introduction

Many decision situations involve outcomes that consist of several attributes. In applied decision analyses it is useful to decompose the utility function over these multiattribute outcomes into separate utility functions over the different attributes so as to reduce the number of preference elicitations. Such decompositions are only justified if the decision maker's preferences satisfy particular assumptions. Several authors have identified the preference conditions that allow decompositions of multiattribute utility functions into additive, multiplicative, and related decompositions (e.g. Farquhar 1975, Fishburn 1965, Keeney and Raiffa 1976).

Most of these decomposition results have been derived under expected utility. Abundant evidence exists, however, that expected utility is not valid as a descriptive theory of decision under uncertainty. The descriptive deficiencies of expected utility complicate the empirical assessment of the preference conditions underlying decompositions: it cannot be excluded that observed violations of preference conditions are due to violations of expected utility rather than to violations of a decomposition. To obtain robust tests of the appropriateness of decompositions, it is desirable to derive conditions that are valid even when expected utility is violated.

In this paper we study multiattribute utility theory under prospect theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992). Prospect theory is currently the most influential theory of decision under uncertainty. It characterizes two major deviations from expected utility: nonlinear decision weighting and loss aversion, i.e. the tendency for people to treat outcomes as deviations from a reference point and be more sensitive to losses than to gains of the same magnitude. Both nonlinear decision weighting and loss aversion are widely documented in the empirical literature.

Fishburn (1984), Miyamoto (1988), Dyckerhoff (1994), and Miyamoto and Wakker (1996) studied multiattribute utility under nonexpected utility, but only considered outcomes of the same sign. Like us, Zank (2001) and Bleichrodt and Miyamoto (2003) studied multiattribute utility theory under prospect theory but their approach was different than the approach of this paper as we will explain next.

A central issue in multi-attribute prospect theory is to determine when an attribute yields a gain or a loss. Consider, for example, a researcher who considers changing jobs. In evaluating different jobs the researcher has to consider several aspects, e.g. salary, commuting time, cost of living, amount of research time, etc. How does the researcher determine whether a particular job offer is an improvement (a gain) compared with his reference point (presumably his current job)? One possibility is that a decision maker first determines whether the job offer as a whole is a gain or a loss compared to his reference point and then applies the decomposition to determine exactly how attractive the job offer is as compared with other offers. This *holistic approach* was used by Zank (2001) and by Bleichrodt and Miyamoto (2003).

Another approach, which is the focus of this paper, is that the researcher determines a reference point for each attribute and evaluates job offers as gains and losses on each attribute. This *segregated approach* seems plausible when the number of attributes is large and the choice is complex. A decision context where the segregated approach is particularly intuitive is welfare theory: there we are interested in whether each individual's welfare is above some reference level. The segregated approach is common in empirical studies on loss aversion for tradeoffs under certainty and was found to be descriptively accurate (Bateman et al. 1997, Bleichrodt and Pinto 2002, Tversky and Kahneman 1991). For the more common case of uncertainty it has not been analyzed, however. Providing such an analysis is the topic of this paper.

The difference between the holistic and the segregate approach is that in the former loss aversion and decision weighting are attribute-independent, whereas in the latter they depend on the attributes. As we will show in Section 5, the holistic and the segregated approach are in general equivalent only when people behave according to expected utility, i.e. when loss aversion does not affect people's preferences and there is no decision weighting. An example to further clarify the difference between the holistic and the segregated approach to multiattribute utility theory is in Section 3.

This paper gives preference foundations for additive utility under prospect theory and the segregated approach. We restrict our attention to the additive decomposition for two reasons. First, it is commonly applied in many areas of economics and decision analysis and, second, other decompositions, such as the multiplicative and the multilinear decompositions raise special problems under the segregated approach. Solving these problems requires different tools and is beyond the scope of the present paper.

The rest of the paper is organized as follows. Section 2 gives notation and explains prospect theory for single-attribute outcomes. We then move to multiattribute utility where we first assume, for ease of exposition, that there are just two attributes, which are both numerical. Section 4 gives preference foundations for prospect theory with additive utility under the segregated approach. As mentioned, weighting functions are defined per attribute and they may differ across attributes in the segregated approach. To force them to be equal across attributes requires additional conditions. We will characterize this special case in Section 5. We extend our results to the case where there are more than two attributes in Section 6 and to the case of nonnumeric outcomes in Section 7. Section 8 concludes the paper with some observations on the empirical measurement of additive utility in prospect theory under the segregated approach. All proofs are in the appendix.

2 Prospect Theory for Single-attribute Outcomes

We consider a decision maker in a situation where there is a finite number, $n \geq 2$ of distinct states of nature, exactly one of which obtains. $S = \{1, ..., n\}$ denotes the set of *states of nature*. Subsets of S are called *events*. In a medical decision problem, the states of nature can, for example, be mutually exclusive diseases and the decision maker has to choose between different treatments before knowing what the actual disease is. We consider decision under uncertainty where the probabilities for the states of nature may, but need not be given. The assumption of a finite number of states of nature is made for expositional purposes. The results of this paper can be extended to an infinite state space using tools from Wakker (1993). The extension to decision under risk, i.e. the case where probabilities are objectively given, is as in Köbberling and Wakker (2003, Section 5.3).

The decision maker's problem is to choose between prospects. Each *prospect* is an n-tuple of *outcomes*, one for each state of nature. Formally, a prospect is a function from the set of states of nature to the set of outcomes C. We denote the set of prospects as $P = C^n$. We shall write $(f_1, ..., f_n)$ for the prospect f that gives f_j if state j occurs. A *constant prospect* gives the same outcome for each state of nature. For ease of exposition, we assume in this section that outcomes are one-dimensional. The set of outcomes C is a nondegenerate convex subset of $\mathbb{R}$. Outcomes are defined with respect to a *reference point*. The reference point is a constant prospect, that we will denote as r. We assume that the reference point is fixed, i.e. we restrict attention to preferences with respect to one reference point. Variations in the reference point are analyzed by Schmidt (2003).

Let $\succcurlyeq$ denote a preference relation on the set of prospects. As usual, $\succ$ denotes the asymmetric part of $\succcurlyeq$ (strict preference) and $\sim$ denotes the symmetric part of $\succcurlyeq$ (indifference), and $\preccurlyeq$ and $\prec$ denote reversed preferences. We shall use the same notation for the binary relations on C derived through constant prospects. An outcome $x \succ r$ is a *gain* and an outcome $x \prec r$ is a *loss*.

A prospect f is *rank-ordered* if $f_1 \succcurlyeq \cdots \succcurlyeq f_n$. For each prospect, there exists a permutation ρ such that $f_{\rho(1)} \succcurlyeq \cdots \succcurlyeq f_{\rho(n)}$. For each permutation ρ let $P_\rho = \{f \in P : f_{\rho(1)} \succcurlyeq \cdots \succcurlyeq f_{\rho(n)}\}$. That is, P_ρ is the set of all prospects that are rank-ordered by ρ. If two prospects can be rank-ordered by a common ρ, then they are *comonotonic.* For each event $A \subset S$, the set P^A contains those prospects that yield gains for states in A and no gains for states not in A. We define the set P^A_ρ as the intersection of P^A and P_ρ. Subsets of sets P^A_ρ are *sign-comonotonic.*

A real-valued function $V : P \to I\!R$ *represents* $\succcurlyeq$ on P if for all $f, g \in P$ we have $f \succcurlyeq g$ if and only if (iff) $V(f) \geqslant V(g)$. A function V is a *ratio scale* if it is unique up to unit, i.e., if V can be replaced by U if and only if $U = \sigma V$ for positive σ. A *weighting function* or *capacity* W is a function on 2^S such that $W(\emptyset) = 0$, $W(S) = 1$, and for any two events A and B, if $B \subset A$ then $W(B) \leq W(A)$. W is *strictly increasing* if $W(B) < W(A)$ whenever B is a proper subset of A.

Prospect theory holds if there exists a utility function $U : C \to I\!R$ with $U(r) = 0$ such that prospects $f \in P^A_\rho$ with $A = \{\rho(1), ..., \rho(k)\}$ for some $k \leq n$ are evaluated by

$$PT(f) = \sum_{j=1}^{k} \pi^+_{\rho(j)} U(f_{\rho(j)}) + \sum_{j=k+1}^{n} \pi^-_{\rho(j)} U(f_{\rho(j)}) \tag{1}$$

with

$$\pi^+_{\rho(j)} = W^+(\rho(1), ..., \rho(j)) - W^+(\rho(1), ..., \rho(j-1)) \tag{2a}$$

and

$$\pi^{-}_{\rho(i)} = W^{-}(\rho(j), ..., \rho(n)) - W^{-}(\rho(j+1), ..., \rho(n)), \tag{2b}$$

and choices and preferences correspond with this evaluation. $PT(f)$ denotes the *prospect theory value*, or *PT value* for short, of f, and W^{+} and W^{-} are weighting functions for gains and losses, respectively. We will assume throughout that U is *strictly increasing*, i.e. for all $x, y \in C$, $x > y$ iff $U(x) > U(y)$, and *continuous.* If prospect theory holds then utility is a ratio scale and the weighting functions are uniquely determined.

3 Prospect Theory for Two-attribute Outcomes

From now on $C = C_1 \times C_2$ is a product of two nondegenerate convex subsets of $I\!R$. Hence, we now deal with two product structures: the two-dimensional structure of C and the n-dimensional structure C^n. In what follows, the index i will refer to the attributes and the index j to the states of nature. Hence, f_{ji} denotes the i-th attribute of the outcome that is obtained under state j. *Outcomes* in C will be denoted as $x = (x_1, x_2)$ or as x_1x_2 for short.

Let P_1 denote the set of prospects on C_1^n and P_2 the set of prospects on C_2^n. For a fixed $f_2 \in P_2$, we define a preference relation $\succcurlyeq_1$ on P_1 by $f_1 \succcurlyeq_1 g_1$ iff $f_1f_2 \succcurlyeq g_1f_2$. We will in Section 4 impose a condition that implies that the choice of f_2 is immaterial. By restricting attention to constant prospects in P_1, we can define a preference relation $\succcurlyeq_1$ on C_1. In a similar fashion we can define $\succcurlyeq_2$ on P_2 and on C_2.

A function $U : C \rightarrow I\!R$ is *additive* if $U : x \longmapsto U_1(x_1) + U_2(x_2)$ where U_i is a real-valued function on $C_i, i = 1, 2$. The functions U_1 and U_2 are *joint ratio scales* if U_1 and U_2 can be replaced by V_1 and V_2 if and only if $V_i = \sigma U_i, \sigma > 0$. That is, any common change in unit is allowed.

In the holistic approach, any outcome x that is indifferent to r can also be interpreted as a reference point. Hence, it does not make sense to consider gains or losses on any separate dimension in the holistic approach. What matters in the holistic approach is whether an outcome x is a gain or a loss compared to r (i.e., whether $x \succ r$ or $x \prec r$, respectively).

Under the holistic approach, additive decomposability means that a prospect $f \in P_\rho^A$ with $A = \{\rho(1), ..., \rho(k)\}$ for some $k \leq n$ is evaluated as

$$PT(f) = \sum_{j=1}^{k} \pi_{\rho(j)}^{+}(U_1(f_{\rho(j)1}) + U_2(f_{\rho(j)2})) + \sum_{j=k+1}^{n} \pi_{\rho(j)}^{-}(U_1(f_{\rho(j)1}) + U_2(f_{\rho(j)2})), \quad (3)$$

where the decision weights are defined as in Eqs. (2a) and (2b). The uniqueness results of prospect theory apply, which implies that the attribute utility functions are joint ratio scales and the weighting function is unique. There is only one permutation function that applies to both attributes. In this representation, the decision weight that is assigned to a single-attribute utility function U_i, $i = 1, 2$, depends on whether the entire outcome is a gain or a loss. If an outcome x is a gain then the decision weight π^+ is applied, if it is a loss then π^- is applied. Preference foundations for Eq. (3) were given by Zank (2001) and Bleichrodt and Miyamoto (2003).

The segregated approach evaluates for each attribute separately whether it yields a gain or a loss, i.e. the segregated approach interprets reference-dependence for each attribute separately. We will denote the *reference point on the first attribute* by r_1 and the *reference point on the second attribute* by r_2. $x_1 \in C_1$ is a gain if $x_1 \succ_1 r_1$ and a loss if $x_1 \prec_1 r_1$ and $x_2 \in C_2$ is a gain if $x_2 \succ_2 r_2$ and a loss if $x_2 \prec_2 r_2$. We will assume that preferences are monotonic in each attribute. Then, in contrast with the holistic approach, the reference point will be unique. We further assume that r_1 is an interior point of C_1 and that r_2 is an interior point of C_2. This ensures that C_1 and C_2 both contain outcomes that are gains and outcomes that are losses and that genuine tradeoffs between gains and losses exist for both attributes.

For each prospect f, there exist permutations ρ_1 and ρ_2 such that $f_{\rho_1(1)1} \succcurlyeq \cdots \succcurlyeq f_{\rho_1(n)1}$ and $f_{\rho_2(1)2} \succcurlyeq \cdots \succcurlyeq f_{\rho_2(n)2}$. Let $P_{\rho_1} = \{f \in P : f_{\rho_1(1)1} \succcurlyeq \cdots \succcurlyeq f_{\rho_1(n)1}\}$. That is, P_{ρ_1} is the set of all prospects for which the first attribute is rank-ordered by ρ_1. P_{ρ_2} is defined similarly. For each event $A_1 \subset S$, the set P^{A_1} contains those prospects that yield gains on the first attribute for states in A_1 and no gains on the first attribute for states not in A_1. Similarly, P^{A_2} contains those prospects that yield gains on the second attribute for states in A_2 and no gains on the second attribute for states not in A_2. We define $P^{A_1}_{\rho_1} = P^{A_1} \cap P_{\rho_1}$ and $P^{A_2}_{\rho_2} = P^{A_2} \cap P_{\rho_2}$. Subsets of $P^{A_1}_{\rho_1}$ are said to be *sign-comonotonic on* C_1 and subsets of $P^{A_2}_{\rho_2}$ are said to be *sign-comonotonic on* C_2. Under the segregated approach, a prospect $f \in P^{A_1}_{\rho_1} \cap P^{A_2}_{\rho_2}$ with $A_1 = \{\rho_1(1), ..., \rho_1(k_1)\}$ and $A_2 = \{\rho_2(1), ..., \rho_2(k_2)\}$ for some $k_1, k_2 \leq n$ is evaluated as

$$\begin{aligned} PT(f) &= \sum_{j=1}^{k_1} \pi^+_{\rho_1(j)1} U_1(f_{\rho_1(j)1}) + \sum_{j=k_1+1}^{n} \pi^-_{\rho_1(j)1} U_1(f_{\rho_1(j)1}) \\ &\quad + \sum_{j=1}^{k_2} \pi^+_{\rho_2(j)2} U_2(f_{\rho_2(j)2}) + \sum_{j=k_2+1}^{n} \pi^-_{\rho_2(j)2} U_2(f_{\rho_2(j)2}). \end{aligned} \tag{4}$$

with

$$\pi^+_{\rho_i(j)i} = W_i^+(\rho_i(1), ..., \rho_i(j)) - W_i^+(\rho_i(1), ..., \rho_i(j-1)),\ i = 1, 2 \tag{5a}$$

and

$$\pi^-_{\rho_i(j)i} = W_i^-(\rho_i(j), ..., \rho_i(n)) - W_i^-(\rho_i(j+1), ..., \rho_i(n)),\ i = 1, 2, \tag{5b}$$

and preferences and choices correspond with this evaluation. The functions U_1 and U_2 are strictly increasing and continuous and satisfy $U_1(r_1) = U_2(r_2) = 0$. $\pi^+_{\cdot 1}$ and $\pi^-_{\cdot 1}$ are the decision weights for gains and losses for the first attribute, $\pi_{.2}^{+}$ and $\pi_{.2}^{-}$ are the decision weights for gains and losses for the second attribute, W_1^+ and W_1^- are the weighting functions for gains and losses for the first attribute, and W_2^+ and W_2^- are the weighting functions for gains and losses for the second attribute. The utility functions are joint ratio scales and the attribute weighting functions are unique. A comparison between Eqs. (3) and (4) reveals that the holistic approach and the segregated approach differ both in loss aversion and in decision weighting.

An example may clarify the difference between the holistic and the segregated approach to additive utility. Suppose that a researcher considers a job offer from a university that will take effect in some months time. If the university cannot attract a better candidate in the meantime, the researcher will be appointed as leader of a research group and becomes full professor. However, if the university finds a better candidate, the researcher will be appointed as member of the research group and becomes assistant professor. A full professor earns \$60K per year and has 15 hours research time per week. An assistant professor earns \$40K per year and has 30 hours research time per week. The researcher's preferences are monotonic both in money (more money is preferred) and in research time (more research time is preferred). Suppose that the researcher's current job, his reference point, earns \$50K per year and has 20 hours research time per week. Suppose also that $(60K, 15h) \succ (50K, 20h) \succ (40K, 30h)$. The researcher's reference point is $(50K, 20h)$ in the holistic approach and in the segregate approach the researcher's reference point for annual earnings is \$50K and for research time is 20 hours per week. In the holistic approach, where we determine first the sign of an outcome and then apply the decompositions, we assume that the researcher's utility function for gains is $u(x_1x_2) - u(r_1r_2)$ and his utility function for losses is $\lambda(\ u(x_1x_2) - u(r_1r_2))$ where λ is a coefficient reflecting loss aversion and u is a *basic utility function* which expresses the researcher's attitude towards outcomes and which is reference independent (Tversky and Kahneman 1991, Köbberling and Wakker 2005). In the segregate approach, where first the decomposition is applied and then it is determined whether attributes yield gains or losses, the utility for gains is $u_i(x_i) - u_i(r_i)$ and for losses it is $\lambda_i(u_i(x_i) - u_i(r_i))$, $i = 1, 2$. We assume that the holistic basic utility is additive such that $u(x_1x_2) = u_1(x_1) + u_2(x_2)$.

The researcher does not care about other job aspects than wage rate and available research time. If event 1 is "no better candidate is found in the meantime" and event 2 is "a better candidate is found in the meantime" then, according to the holistic approach (Eq. (3)), the PT value of this job offer is equal to

$$\begin{aligned} &\pi_1^+((u_1(60) + u_2(15)) - (u_1(50) + u_2(20))) \\ &+\pi_2^-\lambda((u_1(40) + u_2(30)) - (u_1(50) + u_2(20))) \end{aligned} \tag{6}$$

and according to the segregated approach (Eq. (4)), it is equal to

$$\begin{aligned} &\pi_{11}^+(u_1(60) - u_1(50)) + \pi_{12}^-\lambda_2(u_2^-(15) - u_2(20)) \\ &+\pi_{21}^-\lambda_1(u_1(40) - u_1(50)) + \pi_{22}^+(u_2(30) - u_2(20)) \end{aligned} \tag{7}$$

A comparison between Eqs. (6) and (7) shows that both decision weighting and loss aversion differ between the two approaches. While loss aversion and decision weighting are common for single attributes in the holistic approach, they may differ in the segregated approach. First, we may have different decision weighting for the single attributes when the rank-order of outcomes is not identical for both attributes. Second, even if the rank-order is identical, loss aversion and decision weighting may differ as the segregated approach, in general, allows for different degrees of loss aversion and different weighting functions for the single attributes. The possibility of attribute-dependent weighting functions can be realistic in applications. Rottenstreich and Hsee (2001) showed that decision weighting depends on the outcome domain with people deviating more from expected utility for affect-rich outcomes. In Section 5, we will characterize the special case of the segregated approach where the weighting functions are the same across different attributes. There is no empirical evidence to conclude that loss aversion differs across different attributes, but intuitively this seems to make sense.

4 Preference Foundations

This section develops preference conditions for additive prospect theory under the segregated approach, i.e. Eq. (4). We continue to assume that $C = C_1 \times C_2$ with C_1 and C_2 nondegenerate convex subsets of $I\!R$. The preference relation $\succcurlyeq$ on the set of prospects P is a weak order if it is *complete* (for all prospects f, g, $f \succcurlyeq g$ or $g \succcurlyeq f$) and transitive.

Any prospect $f \in P$ yields both a prospect $f_1 \in P_1$ and a prospect $f_2 \in P_2$ and, hence, each prospect f may be viewed as an element of the product $P_1 \times P_2$. Hence, we can denote prospects as $f_1 f_2$. *Weak separability* holds when for all $f_1, g_1 \in P_1$ and for all $f_2, g_2 \in P_2$, $f_1 f_2 \succcurlyeq g_1 f_2$ iff $f_1 g_2 \succcurlyeq g_1 g_2$ and when for all $f_1, g_1 \in P_1$ and for all $f_2, g_2 \in P_2$, $f_1 f_2 \succcurlyeq f_1 g_2$ iff $g_1 f_2 \succcurlyeq g_1 g_2$. Weak separability entails that the relations $\succcurlyeq_1$ on P_1 and $\succcurlyeq_2$ on P_2 are well-defined. *Outcome monotonicity* holds if for $i = 1, 2$, $f_{ji} \geq g_{ji}$ for all j implies $f_i \succcurlyeq_i g_i$ with strict preference holding if one of the antecedent inequalities is strict. *Continuity* holds if for all prospects f_i, the sets $\{g_i \in P_i : g_i \succcurlyeq f_i\}$ and $\{g_i \in P_i : g_i \preccurlyeq f_i\}$ are both closed in $C_i^n, i = 1, 2$.

For $x \in C_i$, $f_i \in P_i$, $i = 1, 2$, and $j \in S$ define

$$x_j f_i = (f_{1i}, ..., f_{j-1i}, x, f_{j+1i}, \dots, f_{ni}),$$

that is, $x_j f_i$ is the prospect f_i with f_{ji} replaced by x. Let $a, b, c, d \in C_1$. We write

$$ab \sim_1^* cd$$

if (i) there exist $f_1, g_1 \in P_1$ and $f_2 \in P_2$ and a state j such that

$$\begin{aligned}(a_j f_1, f_2) &\sim (b_j g_1, f_2) \text{ and}\\ (c_j f_1, f_2) &\sim (d_j g_1, f_2),\end{aligned}$$

where $a_j f_1, b_j g_1, c_j f_1$, and $d_j g_1$ are sign-comonotonic on C_1,

or (ii) there exist $v, w \in C_2$ and $f_1 \in P_1$ such that

$$\begin{aligned}(a_1 f_1, v_1 f_2) &\sim (b_1 f_1, w_1 f_2) \text{ and}\\ (c_1 f_1, v_1 f_2) &\sim (d_1 f_1, w_1 f_2),\end{aligned}$$

where $a_1 f_1, b_1 f_1, c_1 f_1$, and $d_1 f_1$ are rank-ordered prospects in P_1 and $v_1 f_2$, and $w_1 f_2$ are rank-ordered prospects in P_2.

In the first two indifferences the prospect on the second attribute is kept fixed, in the final two indifferences everything outside state of nature 1 is kept fixed. The $\sim_1^*$ relationship may be interpreted as measuring strength of preference. For example, from the indifferences $(a_j f_1, f_2) \sim (b_j g_1, f_2)$ and $(c_j f_1, f_2) \sim (d_j g_1, f_2)$, we can see that $ab \sim_1^* cd$ means that, in the presence of f_2, a tradeoff of a for b is an equally good improvement as a tradeoff of c for d: both exactly offset receiving f_1 instead of g_1 for all other states of nature. A similar interpretation can be assigned to the indifferences $(a_1 f_1, v_1 f_2) \sim (b_1 f_1, w_1 f_2)$ and $(c_1 f_1, v_1 f_2) \sim (d_1 f_1, w_1 f_2)$. Even though the $\sim_1^*$ relations have a natural interpretation in terms of strength of preference, they are defined entirely in terms of observed indifferences and no new primitives beyond observed choice are assumed in their definition. Hence, we stay entirely within the revealed preference paradigm when using the $\sim_1^*$ relations.

Let $w, x, y, z \in C_2$. We define

$$wx \sim_2^* yz$$

if (i) there exist $f_2, g_2 \in P_2$ and $f_1 \in P_1$ and a state j such that

$$\begin{aligned}(f_1, w_j f_2) &\sim (f_1, x_j g_2) \text{ and} \\ (f_1, y_j f_2) &\sim (f_1, z_j g_2),\end{aligned}$$

where $w_j f_2, x_j g_2, y_j f_2$, and $z_j g_2$ are sign-comonotonic on C_2,

or (ii) there exist $a, b \in C_1$ and $f_2 \in P_2$ such that

$$\begin{aligned}(a_1 f_1, w_1 f_2) &\sim (b_1 f_1, x_1 f_2) \text{ and} \\ (a_1 f_1, y_1 f_2) &\sim (b_1 f_1, z_1 f_2),\end{aligned}$$

where $w_1 f_2, x_1 f_2, y_1 f_2$, and $z_1 f_2$ are rank-ordered prospects in P_2 and $a_1 f_1$, and $b_1 f_1$ are rank-ordered prospects in P_1.

We say that $\succcurlyeq$ satisfies *tradeoff consistencyon* C_1 if improving the first attribute of an outcome in any $\sim_1^*$ relationship breaks that relationship. That is, if $ab \sim_1^* cd$ and $a' \succ_1 a$ then it cannot be that $a'b \sim_1^* cd$. Loosely speaking, tradeoff consistency on C_1 ensures that the $\sim_1^*$ relationship is well-behaved when interpreted as a strength of preference relationship. If the strength of preference of a over b is equal to the strength of preference of c over d, then the strength of preference of a' over b cannot be equal to the strength of preference of c over d, when a' is strictly better than a.

Similarly, $\succcurlyeq$ satisfies *tradeoff consistency on* C_2 if improving the second attribute of an outcome in any $\sim_2^*$ relationship breaks that relationship. That is, if $wx \sim_2^* yz$ and $y' \succ_2 y$ then it cannot be that $wx \sim_2^* y'z$. *Tradeoff consistency* holds if tradeoff consistency holds both on C_1 and on C_2. An important advantage of tradeoff consistency as a preference condition is that it is closely related to measurements of utility by the tradeoff method

(Wakker and Deneffe 1996). This makes it easy to test tradeoff consistency empirically. Empirical studies that have used the tradeoff method include Abdellaoui (2000), Etchart-Vincent (2004), Schunk and Betsch (2006), and Abdellaoui, Barrios and Wakker (2007) amongst others.

Solvability holds if for any two prospects $f, g \in P$ there exists outcomes α and β such that $(\alpha_1 f_1, f_2) \sim g$ and $(f_1, \beta_1 f_2) \sim g$. Solvability implies that the attribute utility functions U_1 and U_2 are unbounded.

The next theorem characterizes Eq. (4).

THEOREM 1 *The following two statements are equivalent:*

(i) $\succcurlyeq$ *is represented by the functional in Eq.(4) with strictly increasing weighting functions* W_1^+, W_1^-, W_2^+, *and* W_2^- *and continuous, strictly increasing utility functions* U_1 *and* U_2.

(ii) $\succcurlyeq$ *satisfies (1) weak ordering, (2) continuity, (3) weak separability, (4) outcome monotonicity, (5) solvability, and (6) tradeoff consistency.*

The uniqueness results of prospect theory apply, that is, the weighting functions W_i^+ *and* W_i^-, $i = 1, 2$, *are uniquely determined, and the utility functions* U_1 *and* U_2 *are joint ratio scales.* □

5 Common Weighting Functions

In the segregated approach the weighting functions may differ across the two attributes. In some cases, however, it might be reasonable to take the weighting functions as independent of the attributes. Empirical evidence suggests, for example, that decision weights for money and for health are close (Abdellaoui 2000 compared with Bleichrodt and Pinto 2000). Using common weighting functions facilitates the use of prospect theory in practical applications, because fewer elicitations are required. In this section we will give a preference foundation for the special case of Eq. (4) where the weighting functions do not depend on the attributes.

By continuity and connectedness of C_1 and C_2, there exist gains $x_1 \in C_1$ and $x_2 \in C_2$ and losses $y_1 \in C_1$ and $y_2 \in C_2$ such that $(x_1, r_2) \sim (r_1, x_2)$ and $(y_1, r_2) \sim (r_1, y_2)$ and, hence, such that $U_1(x_1) = U_2(x_2)$ and $U_1(y_1) = U_2(y_2)$. Recall that r is the constant prospect that gives (r_1, r_2) in every state of nature. For any event B, let $x_B f$ denote the prospect f with f_j replaced by x for all j in B. We can now define a condition that ensures attribute independence of the weighting functions for gains and for losses. We say that $\succcurlyeq$ satisfies *attribute-independence for states*, if for all $x_1 \in C_1$ and $x_2 \in C_2$ for which $(x_1, r_2) \sim (r_1, x_2)$ and for all events B, $(x_1, r_2)_B r \sim (r_1, x_2)_B r$. Note that the condition holds for all $x_1 \in C_1$ and $x_2 \in C_2$, but x_1 and x_2 must be either both gains or both losses for otherwise the indifference $(x_1, r_2) \sim (r_1, x_2)$ cannot obtain.

Let us now explain the idea behind the condition. As mentioned before, if $(x_1, r_2) \sim (r_1, x_2)$ then $U_1(x_1) = U_2(x_2)$. If Eq. (4) holds and x_1 and x_2 are both gains, the indifference $(x_1, r_2)_B r \sim (r_1, x_2)_B r$ implies that $W_1^+(B)U(x_1) = W_2^+(B)U(x_2)$ and $W_1^+(B) = W_2^+(B)$ follows from $U_1(x_1) = U_2(x_2)$. A similar line of argument shows that $W_1^-(B) = W_2^-(B)$ whenever x_1 and x_2 are losses. Because these equalities hold for all events B, we obtain the following result.

THEOREM 2 *If we add attribute-independence for states to statement (ii) of Theorem 1 then the weighting functions W^+ and W^- in statement (i) of Theorem 1 are attribute-independent, i.e. for all events E, $W_1^+(E) = W_2^+(E) = W^+(E)$ and $W_1^-(E) = W_2^-(E) =$*

$W^{-}(E)$. □

If $\rho_1 = \rho_2$ and $A_1 = A_2$ then Theorem 2 also implies that the decision weights π^+ and π^- are attribute-independent. This follows straightforwardly from the definition of the decision weights, Eqs. (5a) and (5b). Having the weighting functions independent of the attributes does not make the segregated approach equal to the holistic approach. This is easily seen by referring back to the example of the researcher considering a job offer. Under the segregated approach with common weighting functions, Eq. (7) becomes

$$\pi_1^+(u_1(60) - u_1(50)) + \pi_1^- \lambda_2(u_2(15) - u_2(20)) +$$

$$\pi_2^- \lambda_1(u_1(40) - u_1(50)) + \pi_2^+(u_2(30) - u_2(15)),$$

which clearly differs from the evaluation under the holistic approach, Eq.(6).

Note that it is not only the presence of the loss aversion parameter which distinguishes the holistic from the segregated approach. In general, the two approaches differ even if a prospect yields only gains or only losses. Consider again the job offer example but suppose now that the researcher's reference point for annual earnings is \$30K and for research time it is 10 hours per week. The preference $(60K, 15h) \succ (40K, 30h)$ still holds. Let E_1 denote the first event and E_2 the second event. Then the job offer is evaluated under the holistic approach as

$$W^+(E_1)((u_1(60) + u_2(15)) - (u_1(30) + u_2(10)) +$$

$$(1 - W^+(E_1)((u_1(40) + u_2(15)) - (u_1(30) + u_2(10))$$

and under the segregated approach as

$$W^+(E_1)(u_1(60) - u_1(30)) + (1 - W^+(E_2))(u_2(15) - u_2(10)) +$$

$$(1 - W^+(E_1)(u_1(40) - u_1(30)) + W^+(E_2)(u_2(30) - u_2(10)).$$

Equality only holds if $W^+(E_1) = (1 - W^+(E_2))$, i.e., if $W^+(E_1) + W^+(E_2) = 1$. This must hold for all events E_1 and E_2, which can only be the case if W^+ is a probability measure. A similar argument can be used to derive that W^- must be a probability measure. Hence, for outcomes of the same sign the prospect theory model with attribute independent weighting under the segregated approach agrees with the prospect theory model of the holistic approach only in the case when both representations reduce to subjective expected utility.

6 More than Two Attributes

We will now extend our results to more than two attributes. Let $C = C_1 \times ... \times C_m$, $m > 2$. Each C_i is a nondegenerate convex subset of $I\!R$. The reference point on the i-th attribute is denoted r_i and is assumed to be an interior point of C_i. We will denote the set of prospects on C_i^n as P_i and write prospects as $f_1...f_m$. Let g_if denote the prospect $f \in P$ with f_i replaced by g_i and let g_ih_kf denote the prospect $f \in P$ with f_i replaced by g_i and f_k replaced by h_k. Weak separability is now defined as for all $i \in \{1, ..., m\}$, $f_i, g_i \in P_i$, $f', g' \in P$, $f_if' \succcurlyeq g_if'$ iff $f_ig' \succcurlyeq g_ig'$. The definitions of outcome monotonicity and solvability easily generalize to the case of more than two attributes. For tradeoff consistency we define

$$ab \sim_i^* cd$$

(i) if there exist $f_i, g_i \in P_i$, $f \in P$, and a state j such that

$$(a_jf_i)_if \sim (b_jg_i)_if \text{ and}$$

$$(a_jf_i)_if \sim (b_jg_i)_if,$$

where a_jf_i, b_jg_i, c_jf_i, and d_jg_i are sign-comonotonic on C_i, or (ii) there exist $v, w \in C_k$, and $f \in P$ such that

$$(a_1 f_i)_i (v_1 f_k)_k f \sim (b_1 f_i)_i (w_1 f_k)_k f \text{ and}$$

$$(c_1 f_i)_i (v_1 f_k)_k f \sim (d_1 f_i)_i (w_1 f_k)_k f,$$

where $a_1 f_i, b_1 f_i, c_1 f_i$, and $d_1 f_i$ are rank-ordered prospects in P_i and $v_1 f_k$, and $w_1 f_k$ are rank-ordered prospects in P_k.

Tradeoff consistency holds if each $\sim_i^*$-relationship satisfies tradeoff consistency on C_i.

We are now in a position to extend Theorem 1 to the case of more than two attributes.

THEOREM 3 *The following two statements are equivalent:*

(i) $\succcurlyeq$ *is represented by* $V = \sum_{i=1}^{m} V_i(f_i)$ *where the* V_i *are prospect theory functionals with strictly increasing weighting functions* W_i^+ *and* W_i^- *and continuous, strictly increasing utility functions* U_i*.*

(ii) $\succcurlyeq$ *satisfies (1) weak ordering, (2) continuity, (3) weak separability, (4) outcome monotonicity, (5) solvability, and (6) tradeoff consistency.*

The uniqueness results of prospect theory apply, that is, the weighting functions W_i^+ *and* W_i^- *are uniquely determined, and the utility functions* U_i *are joint ratio scales.*

□

Attribute independence can easily be extended to the case of more than two attributes, so that the arguments preceding Theorem 2 can still be used to ensure that the weighting functions are attribute-independent.

7 General Outcomes

For ease of exposition, we have assumed thus far that all attributes are numerical. In many real-world decisions, this assumption is too restrictive. An example is health, the area in which decision analysis is most frequently applied (Keller and Kleinmuntz 1998, Smith and von Winterfeldt 2004). Health consists of two dimensions, survival duration and health quality, and health quality is a nonnumeric attribute. The extension of our analysis to nonnumeric attributes is straightforward.

Assume that the C_i are connected topological spaces. $C = C_1 \times ... \times C_m$ is endowed with the product topology and so is C^n. The reference points r_i are in the interior of C_i for each i. Redefine outcome monotonicity as for all i, if $f_{ji} \succcurlyeq g_{ji}$ for all j then $f_i \succcurlyeq_i g_i$. The strict version of outcome monotonicity is not necessary here as it follows from the version with weak preferences and tradeoff consistency (Köbberling and Wakker 2003, Lemma 26). We can now state the extension of our results to nonnumeric attributes.

Corollary 4 *If the C_i, $i = 1, ..., m$, are connected topological spaces, then Theorems 1, and 3 still hold if we drop in (i) the requirement that the attribute-wise utility functions are strictly increasing.* □

The proof of this claim follows easily from the proofs of Theorems 1 and 3. Theorem 2 can still be used to ensure that the weighting functions are attribute-independent.

8 Empirical Measurement

Let us finally say a few words about the empirical implementability of additive prospect theory under the segregated approach. The first step is obviously the verification of the preference conditions that have been identified in this paper. When these are satisfied the prospect theory functional must be assessed for each attribute, unless attribute-independence for states holds because in that case the weighting functions need to be assessed only once. Simultaneous measurement of the utility for gains and losses is difficult. A procedure to achieve this was proposed by Abdellaoui, Bleichrodt, and Paraschiv (2007). Their method requires, however, that an "ethically neutral event" exists, i.e. an event that has decision weight 0.5. Hence, we need to impose some richness on the state space to be able to apply this method. When the state space is not sufficiently rich, the method of Abdellaoui, Bleichrodt, and l'Haridon (forthcoming) can be used. This method assumes that the utility functions are power functions.

Measurement of the weighting functions W_i^+ and W_i^-, $i \in \{1, ..., m\}$, can be done either by non choice-based method like in Tversky and Fox (1995), Fox and Tversky (1998), Wu and Gonzalez (1999) and Kilka and Weber (2001) or by choice-based methods as in Abdellaoui, Vossmann, and Weber (2005).

Appendix

Proof of Theorem 1: That (i) implies (ii) is routine. Hence, we assume (ii) and derive (i).

By weak order, weak separability, outcome monotonicity and continuity, $\succcurlyeq$ on P can be represented by $V(V_1(f_1), V_2(f_2))$ with V strictly increasing in V_1 and V_2. V_1 represents $\succcurlyeq_1$ and V_2 represents $\succcurlyeq_2$. By continuity V_1 and V_2 are continuous, by outcome monotonicity, they are strictly increasing.

We will now show that V_1 and V_2 are prospect theory functionals. For a prospect $f_1 \in P_1$, define the prospect f_1^+ by $f_{1j}^+ = f_{1j}$ if $f_{1j} \succ_1 r_1$ and by $f_{1j}^+ = r_1$ otherwise, and the prospect f_1^- by $f_{1j}^- = f_{1j}$ if $f_{1j1} \prec r_1$ and by $f_{1j}^- = r_1$ otherwise. That is, f_1^+ is the positive part of f_1 and f_1^- is its negative part. In a similar fashion we define f_2^+ and f_2^-. Consider $\succcurlyeq_1$ on P_1. Because $\succcurlyeq$ satisfies outcome monotonicity and C_1 is nondegenerate, all states of nature are nonnull (a state is null if replacing any outcomes in that state does not affect the preference). Also, because r_1 lies in the interior of C_1, $\succcurlyeq_1$ is truly mixed ($\succcurlyeq_1$ is *truly mixed* if there exists a prospect f_1 such that $f_1^+ \succ r_1$ and $f_1^+ \prec r_1$, that is, genuine tradeoffs between gains and losses occur). By Theorem 12 in Köbberling and Wakker (2003) there exists a prospect theory representation for $\succcurlyeq_1$ with U_1 the continuous utility function over C_1, $U_1(r_1) = 0$, and W_1^+ and W_1^- the weighting functions over gains and losses on the first attribute, respectively. Köbberling and Wakker's (2003) weak monotonicity follows from outcome monotonicity and sign-comonotonic tradeoff consistency follows from tradeoff consistency on C_1. By Proposition 8.2 in Wakker and Tversky (1993), gain-loss consistency can be dropped from Köbberling and Wakker's (2003) conditions when the number of states of nature exceeds 2. This is the case in our analysis if we interpret attributes as events (Sarin and Wakker 1998, Corollary B.3). U_1 is strictly increasing because V_1 is strictly increasing. W_1^+ and W_1^- are strictly monotone by outcome monotonicity. By Observation 13 in Köbberling and Wakker (2003) U_1 is a ratio scale and W_1^+ and W_1^- are unique. By solvability, U_1 is unbounded.

By a similar line of argument there exists a prospect theory representation for $\succcurlyeq_2$ with U_2 the continuous and strictly increasing utility function on C_2, $U_2(r_2) = 0$, U_2 a ratio scale and W_2^+ and W_2^- the unique and strictly increasing weighting functions over gains and losses on the second attribute, respectively. By solvability, U_2 is unbounded.

So far we have shown that $V(PT_1(f_1), PT_2(f_2))$ represents $\succcurlyeq$. It remains to show that V is additive. We will do so by showing that the rate of trade-off between PT_1

and PT_2 is everywhere constant. Take $f_1 \in P_1$, and let f_2 be a rank-ordered prospect in P_2. Take $\alpha_0^1 \in C_2$ such that $\alpha_0^1 \succcurlyeq f_{22}$. Then $(\alpha_0^1)_1 f_2$ is a rank-ordered prospect in P_2. Let g_2 be such that $f_{2j} \succcurlyeq g_{2j}$ for all j with at least one of these preferences strict. By solvability there exists an outcome α_1^1 such that $(f_1, (\alpha_0^1)_1 f_2) \sim (f_1, (\alpha_1^1)_1 g_2)$. By outcome monotonicity $\alpha_1^1 \succ_2 \alpha_0^1$. Next we consider the prospect $(f_1, (\alpha_1^1)_1 f_2)$. By solvability we can find an outcome α_2^1 such that $(f_1, (\alpha_1^1)_1 f_2) \sim (f_1, (\alpha_2^1)_1 g_2)$. Hence, $\alpha_2^1 \alpha_1^1 \sim_2^* \alpha_1^1 \alpha_0^1$. We proceed in this manner to construct a *standard sequence* $\alpha_0^1, \alpha_1^1, \ldots$ on the second attribute for which $\alpha_s^1 \alpha_{s-1}^1 \sim_2^* \alpha_1^1 \alpha_0^1$ for all natural s. It is easily verified that this implies that $PT_2((\alpha_s^1)_1 f_2) - PT_2((\alpha_{s-1}^1)_1 f_2) = PT_2((\alpha_1^1)_1 f_2) - PT_2((\alpha_0^1)_1 f_2)$. Suppose without loss of generality that $PT_2((\alpha_1^1)_1 f_2) - PT_2((\alpha_0^1)_1 f_2) = 1$.

Next we construct a standard sequence $\beta_0^1, \beta_1^1, \ldots$ on the first attribute by eliciting indifferences $((\beta_t^1)_1 f_1, (\alpha_0^1)_1 f_2) \sim ((\beta_{t-1}^1)_1 f_1, (\alpha_1^1)_1 f_2)$, $j = 1, 2, \ldots$, such that all prospects involved are rank-ordered. These indifferences imply that $\beta_t^1 \beta_{t-1}^1 \sim_1^* \beta_1^1 \beta_0^1$ for all natural t and, thus that $PT_1((\beta_t^1)_1 f_1) - PT_1((\beta_{t-1}^1)_1 f_1) = PT_1((\beta_1^1)_1 f_1) - PT_1((\beta_0^1)_1 f_1)$. The indifferences also define a rate of trade-off between PT_1 and PT_2. Let $PT_1((\beta_1^1)_1 f_1) - PT_1((\beta_0^1)_1 f_1) = c$. Then the rate of trade-off between PT_1 and PT_2 is constant for all the points we have elicited thus far. This claim follows from trade-off consistency. By trade-off consistency, we must have $((\beta_1^1)_1 f_1, (\alpha_{s-1}^1)_1 f_2) \sim ((\beta_0^1)_1 f_1, (\alpha_s^1)_1 f_2)$ for any $s = 1, 2, \ldots$. Applying trade-off consistency again implies that we must have $((\beta_t^1)_1 f_1, (\alpha_{s-1}^1)_1 f_2) \sim ((\beta_{t-1}^1)_1 f_1, (\alpha_s^1)_1 f_2)$ for any $s = 1, 2, \ldots ; t = 1, 2, \ldots$. Hence the rate of trade-off between PT_1 and PT_2 is everywhere c.

Next we double the density of the grid $\{\beta_0^1, \beta_1^1, \ldots\} \times \{\alpha_0^1, \alpha_1^1, \ldots\}$ that we constructed above. By continuity of U_2 and connectedness of C_2 we can find an outcome $\alpha_1^{1/2}$ such that $PT_2((\alpha_1^{1/2})_1 f_2) - PT_2((\alpha_0^1)_1 f_2) = 1/2$. Let $\alpha_0^1 = \alpha_0^{1/2}$ and construct a new standard sequence $\alpha_0^{1/2}, \alpha_1^{1/2}, \ldots$ by eliciting indifferences $(f_1, (\alpha_1^1)_1 f_2) \sim (f_1, (\alpha_2^1)_1 g_2)$. It follows from outcome monotonicity that $\alpha_2^{1/2} = \alpha_1^1$ and, hence, in general $\alpha_{2s}^{1/2} = \alpha_s^1, s = 0, 1, \ldots$.

We construct a new standard sequence $\beta_0^{1/2}, \beta_1^{1/2}, \ldots$ on the first attribute by setting $\beta_0^{1/2} = \beta_0^1$ and eliciting indifferences $((\beta_t^{1/2})_1 f_1, (\alpha_0^{1/2})_1 f_2) \sim ((\beta_{t-1}^{1/2})_1 f_1, (\alpha_1^{1/2})_1 f_2), t = 1, 2, \ldots$. We have to show that the rate of trade-off between PT_1 and PT_2 in this new grid $\{\beta_0^{1/2}, \beta_1^{1/2}, ...\} \times \{\alpha_0^{1/2}, \alpha_1^{1/2}, ...\}$ is still constant. For this we have to show that $\beta_{2t}^{1/2} = \beta_t^1, t = 0, 1, \ldots$. We will show that $\beta_2^{1/2} = \beta_1^1$. $\beta_{2j}^{1/2} = \beta_j^1$, for all $j = 0, 1, ...$then follows from the construction of the standard sequence. By the construction of the standard sequence, $((\beta_2^{1/2})_1 f_1, (\alpha_0^{1/2})_1 f_2) \sim ((\beta_1^{1/2})_1 f_1, (\alpha_1^{1/2})_1 f_2)$. By trade-off consistency $((\beta_1^{1/2})_1 f_1, (\alpha_1^{1/2})_1 f_2) \sim ((\beta_0^{1/2})_1 f_1, (\alpha_2^{1/2})_1 f_2) = ((\beta_0^1)_1 f_1, (\alpha_1^1)_1 f_2) \sim ((\beta_1^1)_1 f_1, (\alpha_0^1)_1 f_2)$. By transitivity and because $\alpha_0^1 = \alpha_0^{1/2}$, $((\beta_2^{1/2})_1 f_1, (\alpha_0^1)_1 f_2) \sim ((\beta_1^1)_1 f_1, (\alpha_0^1)_1 f_2)$. By outcome monotonicity $\beta_2^{1/2} = \beta_1^1$. Hence, the rate of trade-off between PT_1 and PT_2 is still constant when we double the density of the grid.

We continue this doubling of density infinitely, creating increasingly fine standard sequences $\alpha_0^{2^{-m}}, \alpha_1^{2^{-m}}, \ldots$ and $\beta_0^{2^{-m}}, \beta_1^{2^{-m}}, \ldots, m = 2, \ldots$. On the resulting increasingly fine grids the rate of trade-off between PT_1 and PT_2 remains constant by a similar proof as for the case where the density of the grid was doubled.

Because U_1 and U_2 are unbounded, for any natural m there can be no $x_1 \in C_1$ and no $x_2 \in C_2$ such that $x_1 \succ \beta_t^{2^{-m}}$ for all t or $x_2 \succ \alpha_s^{2^{-m}}$ for all s . There can also be no outcomes infinitely close to β_0^1 and α_0^1 in the sense that there is always an outcome from the grid that lies between an outcome $x_1 \in C_1$ and β_0^1 and an outcome from the grid that lies between an outcome $x_2 \in C_2$ and α_0^1 when x_1 is unequal to β_0^1 and x_2 is unequal to α_0^1. If $x_2 \neq \alpha_0^1$ then $U_2(x_2) - U_2(\alpha_0^1) = r > 0$ and, hence, there exists a natural number m such that $2^{-m} < r$. By construction there is an element $\alpha_1^{2^{-m}}$ of the grid such that

$\alpha_0^1 \prec \alpha_1^{2^{-m}} \prec x_2$. A similar argument shows that the grid interferes everywhere. Let x_2 and y_2 be two outcomes such that $x_2 \succ y_2$. Suppose that $U_2(x_2) - U_2(y_2) = r > 0$. Then there exists a natural number m such that $2^{-m} < r$ and by construction there is an element $\alpha_s^{2^{-m}}$ of the grid such that $y_2 \prec \alpha_s^{2^{-m}} \prec x_2$.

Finally, because U_1 and U_2 are unbounded there cannot be elements x_1 and x_2 that are so bad that they are never included in any grid. Consider an outcome x_2. Then we can construct a prospect f_2 with $f_{2j} = x_2$ for all j. Because U_2 is unbounded we can construct a prospect g_2 such that $f_{2j} \succ g_{2j}$ for all j. Let $\alpha_0^1 = x_2$ and construct a new grid by eliciting indifferences $(f_1, (\alpha_0^1)_1 f_2) \sim (f_1, (\alpha_1^1)_1 g_2)$,$(f_1, (\alpha_1^1)_1 f_2) \sim (f_1, (\alpha_2^1)_1 g_2)$ etc. This produces a dense grid that includes x_2.

By continuity we can extend the dense grid to all outcomes. Hence, we have shown that on the whole domain the rate of trade-off between PT_1 and PT_2 is constant for rank-ordered prospects. Hence, for rank-ordered prospects $V(PT_1(f_1), PT_2(f_2))$ is additive: $V(f) = PT_1(f_1) + PT_2(f_2)$. Because U_1 and U_2 are continuos and unbounded and C_1 and C_2 are connected we can for any prospects f_1 and f_2 find rank-ordered prospects g_1 and g_2 such that $f_1 \sim_1 g_1$ and $f_2 \sim_2 g_2$. We set $V(PT_1(f_1), PT_2(f_2)) = PT_1(g_1) + PT_2(g_2)$. Finally, we show that $PT_1(f_1) + PT_2(f_2)$ represents $\succcurlyeq$. Suppose that $f \succcurlyeq g$. There are rank-ordered prospects f' and g' such that $f' \sim f$ and $g' \sim g$. By transitivity, $f' \succcurlyeq g'$. Hence, $PT_1(f_1) + PT_2(f_2) = PT_1(f_1') + PT_2(f_2') \geq PT_1(g_1') + PT_2(g_2') = PT_1(g_1) + PT_2(g_2)$. which completes the proof of statement (i).

The uniqueness results follow from the uniqueness results for PT_1 and PT_2 combined with the fact that on each grid the rate of tradeoff between PT_1 and PT_2 must be constant. This completes the proof of Theorem 1. □

Proof of Theorem 3: That (i) implies (ii) is routine. Hence, we assume (ii) and derive (i). The proof is very similar to the proof of Theorem 1 and will not be elaborated here. By weak separability $V(V_1(f_1), \ldots, V_m(f_m))$ with V strictly increasing in each of the V_i represents $\succcurlyeq$. We then use the results of Köbberling and Wakker (2003) to show that each V_i has a prospect theory representation. Finally, we show, exactly as in the proof of Theorem 1, that for all $i, k \in \{1, ..., m\}$, the rate of trade-off between any PT_i and PT_k is constant. This establishes the proof. □

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