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Keywords: Portfolio theory, transaction costs, Harris recurrence, renewal theory

JEL classification: G11, C61

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A renewal theoretic result in portfolio theory under transaction costs with multiple risky assets

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Abstract

We consider a portfolio optimization problem in a Black-Scholes model with n stocks, in which an investor faces both fixed and proportional transaction costs. The performance of an investment strategy is measured by the average return of the corresponding portfolio over an infinite time horizon. At first, we derive a representation of the portfolio value process, whhich only depends on the relative fractions of the total portfolio value that the investor holds in the different stocks. This representation allows us to consider these so-called risky fractions as the decision variables of the investor. We show a certain kind of stationarity (Harris recurrence) for a quite flexible class of strategies (constant boundary strategies). Then, using renewal theoretic methods, we are able to describe the asymptotic return by the behaviour of the risky fractions in a "typical" period between two trades. Our results generalize those of [4], who considered a financial market model with one bond and one stock, to a market with a finite number $n > 1$ of stocks.

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1 Introduction

In this paper we consider a financial market model with one bond S^0 and n stocks $S^1, \dots, S^n$ which follow geometric Brownian motions.
We measure the performance of an investment strategy by its asymptotic return

$$R = \liminf_{t\to\infty} \frac{1}{t} E(\log X_t).$$

In a version without transaction costs, the problem of maximizing R among all trading strategies was solved in [9]. It turned out that the solution can be found quite easily by describing the dynamics of the portfolio value by the fractions of total wealth held in the different stocks instead of using the absolute values. The optimal strategy is to keep these "risky fractions" constantly equal to a certain vector $\hat{\eta}$, which requires continuous trading (of infinite variation on arbitrary small time intervals) to balance out the fluctuations of the stock market. Of course, this kind of behaviour of an investor is not realistic, especially not, if one includes transaction costs, in which case such a strategy would lead to immediate ruin.
To get a more realistic scenario, the basic model as described above was extended by introducing various kinds of transaction costs in a variety of papers, starting point usually being the case of $n = 1$ stock. Costs are often defined in three different ways: proportionally to the volume of trade (proportional costs), proportionally to the portfolio value (fixed costs), or consisting of a combination of proportional and fixed costs. The research has shown that the structure of optimal strategies depends essentially on the type of transaction costs and only to a lesser amount on the optimization criterion.
[8] considered the problem of maximizing a discounted consumption criterion for proportional costs in the case of one stock and made a conjecture for the structure of the optimal solution, which was rigorously proved to be valid in [3]. [12] provide a numerical method to tackle the same problem in the general case with n stocks.
The problem of maximizing the asymptotic return in the case of proportional transaction costs in the one-stock case is examined in [14]. The optimal investment strategy consists in keeping the fraction of wealth invested in the stock in a fixed interval $[a, b]$. The investor does not trade at all, as long as the risky fraction is in (a, b). Whenever the risky fraction hits the boundary of this interval, infinitesimal trading occurs, which just keeps the risky fraction inside the

interval. [1]study the corresponding problem in the n-stock case. In this paper, the problem is traced back to a variational inequality, and a numerical solution is provided for the case of $n = 2$ stocks.
For fixed costs, an optimal strategy for the problem of maximizing the asymptotic growth rate in the one-stock case is given in [11]. Again, there is an interval (a, b) for the risky fraction, where it is optimal not to trade at all. But when the boundary of this interval is reached, due to the fixed costs the investor chooses a certain point (constant over time) inside the interval as the new risky fraction. [2] perform an asymptotical analysis (around the solution without transaction costs) in the n-stock case to get approximate solutions.
[4] consider the one-stock case under a combination of fixed and proportional costs. In this paper the analysis is restricted to so called constant boundary strategies, which can be described by four parameters $a < \alpha < \beta < b$. The investor does not trade at all, as long as the risky fraction is in the interval (a, b). When it hits the boundary at a resp. b, he chooses the new risky fraction as α resp. β. It is shown in the paper that the asymptotic return of such a strategy can be traced back to the behaviour of the risky fraction in a "typical" period between two trades. This result is then used to get an explicit expression for the asymptotic return, which allows a (nearly) explicit computation of the optimal strategy within the class of constant boundary strategies. [5] show that this strategy is in fact even optimal in the class of all trading strategies.
[7] treats the general case with n stocks under fixed and proportional costs for the problem of maximizing a discounted consumption criterion and gives a solution for the case of uncorrelated stock returns.
In this paper we consider a natural generalization of constant boundary strategies to the case of n stocks, and we the extend the result of [4], which traces back the asymptotic return to a typical period between two trades, to this general case. The main result is stated as theorem 3.7.
In section 2 we introduce the model with fixed and proportional transaction costs and show that there is a one-to-one correspondence between impulse control strategies (specifying the absolute values of a transaction) and new risky fraction strategies (specifying the fractions of wealth invested in the different stocks after a transaction). Furthermore, we provide a factorization of the wealth into the wealth gained in the periods between two trades, which leads to an additive structure of the logarithmized wealth. In section 3 we introduce constant boundary strategies and use the additive structure of the logarithmized wealth to adapt renewal theoretic methods to the logarithmized wealth process, which allows us to trace back the asymptotic return to a typical period between two trades.

2 Fixed and proportional transaction costs

We consider a financial market model with one bond S^0 and n stocks $S^1, \ldots, S^n$. The prices $(S_t)_{t\in[0,\infty)} = ((S_t^0, \ldots, S_t^n)^T)_{t\in[0,\infty)}$ are given by the SDEs

$$\begin{aligned} dS_t^0 &= rS_t^0 dt, \\ dS_t^i &= \mu_i S_t^i dt + \sum_{j=1}^n \sigma_{ij} S_t^i dW_t^j,\ i \in \{1, \ldots, n\}, \end{aligned} \tag{1}$$

with starting values $S_0^i = 1$, interest rate $r \in \mathbb{R}$, trend $\mu = (\mu_1, \ldots, \mu_n)^T \in \mathbb{R}^n$, positive definite volatility matrix $\sigma = \{\sigma_{ij}\}_{i,j\in\{1,\ldots,n\}}$ and an n-dimensional Brownian motion $W = (W^1, \ldots, W^n)^T$ with standard filtration $(\mathcal{F}_t)_t$.
An investor faces a combination of costs proportional to the portfolio value (fixed costs) and costs proportional to the trading volume in each stock (proportional costs). The cost function c is therefore given by

$$c : (0, \infty) \times \mathbb{R}^n \mapsto (0, \infty),\ (x, \Delta) \mapsto \delta x + \sum_{i=1}^n \gamma_i |\Delta_i|, \tag{2}$$

where $\delta \in (0, 1),\ \gamma_1, \ldots, \gamma_n \in [0, 1 - \delta)$. δ denotes the fraction of the portfolio value and γ_i the fraction of the transaction volume in the i-th stock that has to be paid for every transaction.
We do not allow short selling, so the range of admissible fractions of the total portfolio value held in the stocks is given by

$$\mathcal{M} = \{\pi = (\pi^1, \ldots, \pi^n)^T \in [0, 1]^n : \sum_{i=1}^n \pi^i \leq 1\}. \tag{3}$$

By the definition of c we immediately get

$$|c(x, \Delta) - c(x, \Delta')| < \sum_{i=1}^n |\Delta_i - \Delta_i'| \tag{4}$$

for all $x > 0, \Delta, \Delta' \in \mathbb{R}^n$ with $\Delta \neq \Delta'$ and

$$c(x, -\pi^1 x, \ldots, -\pi^n x) < x \tag{5}$$

for all $x > 0$, $\pi \in \mathcal{M}$.

From the fractions of wealth held in the stocks we can directly calculate the fraction of wealth held in the bond, so the following notation will be useful: For every $\pi = (\pi^1, \dots, \pi^n)^T \in \mathcal{M}$ let

$$\pi^0 = 1 - \sum_{i=1}^{n} \pi^i. \tag{6}$$

As the cost structure in our model includes a fixed component, infinitesimal trading of an investor automatically leads to immediate ruin. So it is sensible to describe the trading of an investor by an increasing sequence of stopping times (the times when trading occurs) and a sequence of $\mathbb{R}^n$-valued random variables, whose i-th entries are the amounts of money, for which the i-th stock is bought (positive sign) or sold (negative sign) at the trading times, see e.g. [11]. Such strategies are often called impulse control strategies.
We are now in the position to introduce the most important notions to formalize the trading in our model. This is done as in [4], [5].

2.1 Definition
(i) An impulse control strategy $\tilde{K} = (\tau_k, \Delta_k)_{k\in\mathbb{N}_0}$ is given by stopping times $0 = \tau_0 \le \tau_1 \le \ldots \le \infty$ with $P(\tau_k \to \infty) = 1$ and by $\mathcal{F}_{\tau_k}$-measurable $\mathbb{R}^n$-valued random variables $(\Delta_k)_{k\in\mathbb{N}_0}$.

(ii) In the following we consider a fixed starting wealth $X_0 = x > 0$. We define $(0, \infty)$-valued processes

- $(X_t)_{t\in[0,\infty)}$ (wealth process),
- $(V_k)_{k\in\mathbb{N}_0}$ (new wealth process)

and $\mathcal{M}$-valued processes

- $(\pi_t)_{t\in[0,\infty)} = ((\pi_t^1, \dots, \pi_t^n)^T)_{t\in[0,\infty)}$ (risky fraction process),
- $(\eta_k)_{k\in\mathbb{N}_0} = ((\eta_k^1, \dots, \eta_k^n)^T)_{k\in\mathbb{N}_0}$ (new risky fraction process)

by setting for all $k \in \mathbb{N}_0$ on $\{\tau_k < \infty\}$

$$V_k = X_{\tau_k} - c(X_{\tau_k}, \Delta_k), \tag{7}$$

$$\eta_k^i = \frac{\pi_{\tau_k}^i X_{\tau_k} + \Delta_k^i}{V_k}, \; i \in \{1, \dots, n\}, \tag{8}$$

$$X_t = \sum_{i=0}^{n} \eta_k^i \frac{S_t^i}{S_{\tau_k}^i} V_k, \; t \in (\tau_k, \tau_{k+1}], \tag{9}$$

$$\pi_t^i = \frac{\eta_k^i V_k S_t^i}{S_{\tau_k}^i X_t}, \; t \in (\tau_k, \tau_{k+1}], \; i \in \{1, \dots, n\}. \tag{10}$$

An impulse control strategy is called admissible, if for all starting values $\pi_0 = \pi \in \mathcal{M}$ of the risky fractions we have $V_t > 0$ and $\pi_t \in \mathcal{M}$ for all $t > 0$.
In the following we denote
$$P_\pi = P(\cdot|\pi_0 = \pi) \tag{11}$$
for all $\pi \in \mathcal{M}$, and we write E_π for the expectation with respect to P_π .
It is frequently useful to indicate which strategy $\tilde{K}$ is used. For this, we write $X_t^{\tilde{K}}$ (instead of X_t).

(iii) For any admissible impulse control strategy $\tilde{K}$ the asymptotic return for a starting risky fraction $\pi \in \mathcal{M}$ is given by
$$R^{\tilde{K}}(\pi) = \liminf_{t\to\infty} \frac{1}{t} E_\pi(\log X_t^{\tilde{K}}). \tag{12}$$
In the portfolio optimization problem of [9] without transaction costs, it proved essential to consider risky fractions instead of the absolute trading volumes as the decision variables of an investor. Motivated by this fact, we shall now describe the trading of an investor by "new risky fraction strategies" (as defined below) instead of impulse control strategies, which will be justified by establishing a one-to-one relationship between these classes of strategies.

2.2 Definition
A New Risky Fraction strategy (NRF strategy) $(\tau_k, \eta_k)_{k\in\mathbb{N}_0}$ is given by stopping times $0 = \tau_0 \le \tau_1 \le \ldots \le \infty$ with $P(\tau_k \to \infty) = 1$, and by $\mathcal{F}_{\tau_k}$-measurable $\mathcal{M}$-valued random variables η_k.

The following lemma is the main step to prove the one-to-one relationship of NRF strategies and impulse control strategies. It basically states that given any portfolio value x and any vector of risky fractions π, the absolute trading volumes of a transaction can be recovered by the new risky fractions (after the transaction) and vice versa. This is easily shown in the case of only one stock (see [4], Lemma 3.5), but it is non-trivial in the general case.

2.3 Lemma
Let $x > 0$ and $\pi \in \mathcal{M}$.
(i) Let
$$\begin{aligned}\mathcal{D}(x,\pi) &= \{\Delta \in \mathbb{R}^n : \pi^i x + \Delta^i \ge 0 \text{ for all } i \in \{1,\ldots,n\},\\ &\qquad \pi^0 x - c(x,\Delta) - \sum_{i=1}^n \Delta^i \ge 0\}.\end{aligned} \tag{13}$$

Then $\mathcal{D}(x,\pi) \neq \emptyset$, and for all $\Delta \in \mathcal{D}(x,\pi)$ we have $x - c(x,\Delta) > 0$.
An impulse control strategy $(\tau_k, \Delta_k)_{k\in\mathbb{N}_0}$ is admissible, if and only if for all $k \in \mathbb{N}_0$ we have $\Delta_k \in \mathcal{D}(X_{\tau_k}, \pi_{\tau_k})$.
(ii) The mapping

$$f_{x,\pi} : \mathcal{D}(x,\pi) \mapsto \mathcal{M},\ \Delta \mapsto (\frac{\pi^1 x + \Delta^1}{x - c(x,\Delta)}, \ldots, \frac{\pi^n x + \Delta^n}{x - c(x,\Delta)})^T. \tag{14}$$

is a bijection.

Proof:
(i) Note first that for all $\Delta \in \mathbb{R}^n$ it holds that

$$x - c(x,\Delta) = \sum_{i=1}^{n} (\pi^i x + \Delta^i) + (\pi^0 x - c(x,\Delta) - \sum_{i=1}^{n} \Delta^i). \tag{15}$$

Let $\overline{\Delta} = (-\pi^1 x, \ldots, -\pi^n x)^T$. By (5) we get $x - c(x, \overline{\Delta}) > 0$, so (15) shows $\overline{\Delta} \in \mathcal{D}(x,\pi)$, hence $\mathcal{D}(x,\pi) \neq \emptyset$.
Now let $\Delta \in \mathcal{D}(x,\pi)$. By (13), all $n+1$ addends on the right hand side are ≥ 0. If all of the first n addends are equal to 0, we have $\Delta = \overline{\Delta}$, hence the $(n+1)$-th addend is > 0, which shows $x - c(x,\Delta) > 0$.
The last claim now follows directly from (7) and (8).

(ii) We denote the i-th coordinate function of $f_{x,\pi}$ by f_i (i.e. $f_{x,\pi} = (f_1, \ldots, f_n)^T$).
At first we show that $f_{x,\pi}$ is injective.
We assume that there exist $\Delta, \tilde{\Delta} \in \mathcal{D}(x,\pi)$ with $f_{x,\pi}(\Delta) = f_{x,\pi}(\tilde{\Delta})$ and $\Delta \neq \tilde{\Delta}$.
Without loss of generality we may assume that there exists an i with $\Delta^i > \tilde{\Delta}^i$.

Case 1: There exists a j with $\Delta^j < \tilde{\Delta}^j$.
Then

$$\begin{aligned} f_i(\Delta) f_j(\tilde{\Delta}) &= \frac{(\pi^i x + \Delta^i)(\pi^j x + \tilde{\Delta}^j)}{(x - c(x,\Delta))(x - c(x,\tilde{\Delta}))} \\ &> \frac{(\pi^i x + \tilde{\Delta}^i)(\pi^j x + \Delta^j)}{(x - c(x,\Delta))(x - c(x,\tilde{\Delta}))} \\ &= f_i(\tilde{\Delta}) f_j(\Delta), \end{aligned} \tag{16}$$

which is a contradiction.

Case 2: It holds $\Delta^j \geq \tilde{\Delta}^j$ for all j.
Then we get (using (4))

$$\begin{aligned}(\sum_{j=1}^n f_j(\Delta))(1-\sum_{j=1}^n f_j(\tilde{\Delta})) &= \frac{(\sum_{j=1}^n(\pi^j x+\Delta^j))(\pi^0 x - c(x,\tilde{\Delta})-\sum_{j=1}^n \tilde{\Delta}^j)}{(x-c(x,\Delta))(x-c(x,\tilde{\Delta}))} \\ &> \frac{(\sum_{j=1}^n(\pi^j x+\tilde{\Delta}^j))(\pi^0 x - c(x,\Delta)-\sum_{j=1}^n \Delta^j)}{(x-c(x,\Delta))(x-c(x,\tilde{\Delta}))} \\ &= (\sum_{j=1}^n f_j(\tilde{\Delta}))(1-\sum_{j=1}^n f_j(\Delta)), \qquad (17)\end{aligned}$$

which is again a contradiction.
Hence $f_{x,\pi}$ is injective.

We now show that $f_{x,\pi}$ is surjective.
Let $b = (b^1, \ldots, b^n)^T \in \mathcal{M}$. As $f_{x,\pi}(\overline{\Delta}) = 0$, we can assume $b \neq 0$ and without loss of generality even $b_1 > 0$.
For $i \in \{1, \ldots, n\}$ let

$$\beta : [-\pi^1 x, \infty) \mapsto \mathbb{R}^n,\ y \mapsto (y, \frac{b^2}{b^1}(y+\pi^1 x) - \pi^2 x, \ldots, \frac{b^n}{b^1}(y+\pi^1 x) - \pi^n x)^T, \quad (18)$$

and $\beta_1, \ldots, \beta_n$ the coordinate functions of β. Furthermore let

$$h : [-\pi_1 x, \infty) \mapsto \mathbb{R},\ y \mapsto \pi^0 x - c(x, \beta(y)) - \sum_{i=1}^n \beta_i(y). \quad (19)$$

h is continuous, strictly decreasing (according to (4)) and, using (5), we get

$$h(-\pi^1 x) > 0,\ \lim_{y\to\infty} h(y) = -\infty. \quad (20)$$

Thus by the intermediate value theorem of elementary calculus there exists a unique $m \in (-\pi^1 x, \infty)$ with $h(m) = 0$, and we have

$$\{y : h(y) \geq 0\} = [-\pi^1 x, m]. \quad (21)$$

As all mappings β_i are strictly increasing, it now directly follows from (13) that

$$\{y : \beta(y) \in \mathcal{D}(x, \pi)\} = [-\pi^1 x, m]. \quad (22)$$

Because of

$$\sum_{i=1}^n f_i(\beta(-\pi^1 x)) = 0,\ \sum_{i=1}^n f_i(\beta(m)) = 1 \quad (23)$$

there exists (by the intermediate value theorem again) $y^* \in [-\pi^1 x, m]$ with

$$\sum_{i=1}^{n} f_i(\beta(y^*)) = \sum_{i=1}^{n} b^i. \tag{24}$$

Then we have $\Delta^* = \beta(y^*) \in \mathcal{D}(x, \pi)$, and the following equations hold:

$$f_i(\Delta^*) = \frac{b^i}{b^1} f_1(\Delta^*) \text{ for all } i \in \{1, ..., n\}, \tag{25}$$

$$\sum_{i=1}^{n} f_i(\Delta^*) = \sum_{i=1}^{n} b^i. \tag{26}$$

This shows $f_i(\Delta^*) = b^i$ for all $i \in \{1, \ldots, n\}$, hence $f_{x,\pi}(\Delta^*) = b$, so the claim is proved.

2.4 Theorem
The mapping

$$\begin{aligned} f : \{\tilde{K} : \tilde{K} \text{ admissible impulse control strategy}\} &\mapsto \{K : K \text{ NRF strategy}\}, \\ \tilde{K} &\mapsto ((\tau_k^{\tilde{K}})_{k \in \mathbb{N}_0}, (\eta_k^{\tilde{K}})_{k \in \mathbb{N}_0}) \end{aligned} \tag{27}$$

is a bijection.

Proof:
Let $K = (\tau_k^K, \eta_k^K)_{k \in \mathbb{N}_0}$ be an NRF strategy, and let $f_{x,\pi}$ be defined as in (14). We define

$$\Delta_0^K = f^{-1}_{X_0, \pi_0}(\eta_0^K), \tag{28}$$

and recursively

$$\Delta_k^K = f^{-1}_{X_{\tau_k^K}, \pi_{\tau_k^K}}(\eta_k^K), \tag{29}$$

where $X_{\tau_k^K}$, $\pi_{\tau_k^K}$ are defined as in (7)-(10) (hence these random variables are uniquely determined by $\Delta_0^K, \ldots, \Delta_{k-1}^K$). By Lemma 2.3 the process

$$\delta(K) = (\tau_k^K, \Delta_k^K)_{k \in \mathbb{N}_0} \tag{30}$$

is an admissible impulse control strategy.
Using the fact that $\eta_k^{\tilde{K}} = f_{X_{\tau_k^{\tilde{K}}}, \pi_{\tau_k^{\tilde{K}}}}(\Delta_k)$ for all admissible impulse control strategies $\tilde{K}$ and all $k \in \mathbb{N}_0$, it follows by induction that the function δ defined by (30) is an inverse function to f.

Theorem 2.4 justifies to use NRF strategies instead of impulse control strategies to describe trading strategies. We will now express the wealth process of a portfolio in terms of the risky fractions.
In the following let for any $t \in [0, \infty)$

$$M_t = \sup\{k \in \mathbb{N}_0 : \tau_k < t\}. \tag{31}$$

2.5 Theorem
Let $(\tau_k, \Delta_k)_{k \in \mathbb{N}_0}$ be an admissible impulse control strategy with $\eta_k^i > 0$ for all $k \in \mathbb{N}_0,\ i \in \{1, \dots, n\}$.
(i) For all $k \in \mathbb{N}_0,\ i \in \{1, \dots, n\}$ we have

$$A_k^i = \operatorname{sign}(\Delta_k^i) = \operatorname{sign}(\eta_k^i(1 - \delta - \sum_{j=1}^n \gamma_j |\frac{\eta_k^j}{\eta_k^i} \pi_{\tau_k}^i - \pi_{\tau_k}^j|) - \pi_{\tau_k}^i). \tag{32}$$

(ii) For all $k \in \mathbb{N}_0$ we have

$$V_k = \frac{1 - \delta + \sum_{i=1}^n \gamma_i A_k^i \pi_{\tau_k}^i}{1 + \sum_{i=1}^n \gamma_i A_k^i \eta_k^i} X_{\tau_k}. \tag{33}$$

(iii) If $\eta_k \in int(\mathcal{M})$ for all $k \in \mathbb{N}_0$ (and hence $\pi_t \in int(\mathcal{M})$ for all $t \in (0, \infty)$), then for all $t \in (0, \infty)$ we have

$$X_t = X_0 e^{rt} \frac{\eta_{M_t}^0}{\pi_t^0} \frac{1 - \delta + \sum_{i=1}^n \gamma_i A_0^i \pi_0^i}{1 + \sum_{i=1}^n \gamma_i A_0^i \eta_0^i} \prod_{k=1}^{M_t} \frac{\eta_{k-1}^0}{\pi_{\tau_k}^0} \frac{1 - \delta + \sum_{i=1}^n \gamma_i A_k^i \pi_{\tau_k}^i}{1 + \sum_{i=1}^n \gamma_i A_k^i \eta_k^i}. \tag{34}$$

Proof:
(i) Let $k \in \mathbb{N}_0,\ i \in \{1, \dots, n\}$. From (7), (8) and (2) it follows that

$$\begin{aligned}
\Delta_k^i &= \eta_k^i(X_{\tau_k}(1 - \delta) - \sum_{j=1}^n \gamma_j |\Delta_k^j|) - \pi_{\tau_k}^i X_{\tau_k} \\
&= \eta_k^i(X_{\tau_k}(1 - \delta) - \sum_{j=1}^n \gamma_j |\frac{\eta_k^j}{\eta_k^i}(\pi_{\tau_k}^i X_{\tau_k} + \Delta_k^i) - \pi_{\tau_k}^j X_{\tau_k}|) - \pi_{\tau_k}^i X_{\tau_k}.
\end{aligned} \tag{35}$$

For any $x > 0, \eta, \pi \in \mathcal{M}$ with $\eta^i > 0$ we now define

$$g_{x,\pi,\eta} : \mathbb{R} \mapsto \mathbb{R},\ z \mapsto \eta^i(x(1 - \delta) - \sum_{j=1}^n \gamma_j |\frac{\eta^j}{\eta^i}(\pi^i x + z) - \pi^j x|) - \pi^i x. \tag{36}$$

A simple calculation shows $g_{x,\pi,\eta}(-\pi^i x) > -\pi^i x$, furthermore $g_{x,\pi,\eta}$ is continuous and concave, and for the right-hand side derivative we have

$$\lim_{z\to\infty} g'_{x,\pi,\eta}(z) < 0. \tag{37}$$

It follows that $g_{x,\pi,\eta}$ has a uniquely determined fixed point z^* on $(-\pi^i x, \infty)$ with

$$\operatorname{sign}(z^*) = \operatorname{sign}(g_{x,\pi,\eta}(0)) = \operatorname{sign}(\eta^i(1-\delta-\sum_{j=1}^{n}\gamma_j|\frac{\eta^j}{\eta^i}\pi^i-\pi^j|)-\pi^i). \tag{38}$$

Using $\Delta_k^i = g_{X_{\tau_k},\pi_{\tau_k},\eta_k}(\Delta_k^i)$ the claim follows.

(ii) From (7), (8) and (2) we get

$$V_k = (1-\delta)X_{\tau_k} - \sum_{i=1}^{n}\gamma_i A_k^i(\eta_k^i V_k - \pi_{\tau_k}^i X_{\tau_k}), \tag{39}$$

solving for V_k proves (33).

(iii) By using (9) and (10) we get

$$X_t = (\eta_{k-1}^0 e^{r(t-\tau_{k-1})} + \sum_{i=1}^{n}\frac{\pi_t^i X_t}{V_{k-1}})V_{k-1} \tag{40}$$

for all $k \in \mathbb{N}$ and $t \in (\tau_{k-1}, \tau_k]$ and hence

$$X_t = \frac{\eta_{k-1}^0}{\pi_t^0} e^{r(t-\tau_{k-1})} V_{k-1}. \tag{41}$$

Plugging into (33) and using induction yields

$$V_{M_t} = V_0 e^{r\tau_{M_t}} \prod_{k=1}^{M_t}(\frac{\eta_{k-1}^0}{\pi_{\tau_k}^0}\frac{1-\delta+\sum_{i=1}^{n}\gamma_i A_k^i \pi_{\tau_k}^i}{1+\sum_{i=1}^{n}\gamma_i A_k^i \eta_k^i}). \tag{42}$$

Now the claim follows from (33) and (41).

We are now in the position to describe (as announced above) the trading in our model by NRF strategies.

2.6 Definition
(i) Let f be the bijection of Theorem 2.4. For a given NRF strategy K we define the risky fraction process $(\pi_t^K)_{t\in[0,\infty)}$, the wealth process $(X_t^K)_{t\in[0,\infty)}$ and the asymptotic return R^K (resp. $R^K(x)$) by the corresponding processes/variables of the impulse control strategy $\tilde{K} = f^{-1}(K)$.
(ii) The type of transaction in stock $i \in \{1, \dots, n\}$ is defined by

$$A^i(\pi, \eta) = \mathrm{sign}(\eta^i(1 - \delta - \sum_{j=1}^{n} \gamma_j |\frac{\eta^j}{\eta^i}\pi^i - \pi^j|) - \pi^i). \tag{43}$$

for all $\pi, \eta \in \mathcal{M}$ with $\eta^i > 0$.
(iii) For all $\pi, \eta \in int(\mathcal{M})$ we define the gain function g by

$$g(\pi, \eta) \;=\; \log(\frac{\eta^0}{\pi^0}) + \log(\frac{1 - \delta + \sum_{i=1}^n \gamma_i A_k^i(\pi, \eta)\pi^i}{1 + \sum_{i=1}^n \gamma_i A_k^i(\pi, \eta)\eta^i}). \tag{44}$$

For the rest of this section we take an NRF strategy with $\eta_k \in int(\mathcal{M})$ for all $k \in \mathbb{N}_0$ as given.

2.7 Remark
Taking logarithms and changing indices in Theorem 2.5 (iii) immediately yields

$$\log X_t = \log X_0 + rt + \log(\frac{\pi_0^0}{\pi_t^0}) + \sum_{k=0}^{M_t} g(\pi_{\tau_k}, \eta_k). \tag{45}$$

If furthermore $\limsup_{t\to\infty} \frac{1}{t} E(\log \pi_t^0) = 0$, it follows that

$$R^K = r + \liminf_{t\to\infty} \frac{1}{t} E(\sum_{k=0}^{M_t} g(\pi_{\tau_k}, \eta_k)). \tag{46}$$

The representation (46) will be the starting point for our considerations in Section 3. Additionally, we need some technical results on the dynamics of the risky fraction process between two transactions.
For this, it proves useful to introduce the wealth process without trading $(\overline{X}_t)_{t\in[0,\infty)}$ and the risky fraction process without trading $(\overline{\pi}_t)_{t\in[0,\infty)}$, which are defined by

$$\overline{X}_t = \sum_{i=0}^{n} \eta_0^i V_0 S_t^i, \quad \overline{\pi}_t^i = \frac{\eta_0^i V_0 S_t^i}{\overline{X}_t}. \tag{47}$$

The dynamics of the wealth process without trading can be described by a system of stochastic differential equations stated in the following lemma, which is the same as Lemma 3.1 of [11].

2.8 Lemma
The process $(\overline{\pi}_t)_{t\in[0,\infty)}$ solves the stochastic differential equations

$$d\overline{\pi}_t^i = (\overline{\pi}_t^i((e_i - \overline{\pi}_t)^T(\mu - r\overline{1} - \sigma\sigma^T\overline{\pi}_t)))dt + (\overline{\pi}_t^i(e_i - \overline{\pi}_t)^T\sigma)dW_t \tag{48}$$

for all $i \in \{1, \dots, n\}$, where e_i denotes the i-th unit vector.
The infinitesimal generator $\mathcal{L}$ of $(\overline{\pi}_t)_{t\in[0,\infty)}$ is thus given by

$$\begin{aligned}\mathcal{L}u(x) \quad &= \quad \sum_{i=1}^n x_i(e_i - x)^T(\mu - r\overline{1} - \sigma\sigma^T x)\frac{\partial u(x)}{\partial x_i} \\ &\quad + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n x_i x_j (e_i - x)^T\sigma\sigma^T(e_j - x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j}\end{aligned} \tag{49}$$

for all $u \in C^2(int(\mathcal{M}))$.

We conclude this section with a technical result, crucially required in Section 3, which states that the infinitesimal generator of the risky fraction process is uniformly elliptic on closed subsets of $int(\mathcal{M})$. We omit the elementary proof.

2.9 Lemma
For all $x \in \mathbb{R}^n$ let

$$\Sigma(x) = (x_i x_j (e_i - x)^T\sigma\sigma^T(e_j - x))_{i,j\in\{1,\dots,n\}}. \tag{50}$$

Let $A \subseteq \mathbb{R}^n$ with $\overline{A} \subseteq int(\mathcal{M})$. Then there exist λ, Λ with $0 < \lambda < \Lambda$, such that for all $x \in A$ and all $v \in \mathbb{R}^n$ we have

$$\lambda v^T v \le v^T\Sigma(x)v \le \Lambda v^T v. \tag{51}$$

3 Constant boundary strategies

In the model without transaction costs the main benefit of considering risky fractions instead of the absolute sizes of transactions was the result that the optimal behaviour of an investor did not change over time. The representation (46) of the asymptotic return given in the last section suggests that a certain kind of

stationarity should also be existent in the model with transaction costs.
Hence, the following behaviour of an investor seems reasonable: Starting with a favourable portfolio, the investor intervenes, whenever the risky fractions have (caused by the fluctuations of the stock prices) left a fixed set (of favourable risky fractions) A. The portfolio has then become just unfavourable enough that he accepts the costs for a transaction to rearrange his portfolio. He then chooses his new portfolio in such a way that the new risky fractions are in a subset B of the favourable set A. As our cost structure includes proportional costs, his choice $\phi(x)$ will depend on the point $x \in \partial A$, where the risky fraction process leaves A. Starting with the chosen new risky fractions the investor now iterates this procedure.
Any of such so called constant boundary strategies is determined by the sets A, B and the "choice function" $\phi : \partial A \mapsto B$. The examination of constant boundary strategies is additionally motivated by the fact that (as mentioned in the introduction) the solution to the problem of maximizing the asymptotic return in the special case of only one stock can be found in this class of strategies as was shown in [5].
We start by giving a formal definition of constant boundary strategies.

3.1 Definition
An NRF-strategy $(\tau_k, \eta_k)_{k\in\mathbb{N}_0}$ is called constant boundary strategy (CB strategy), if there exist an open, connected set A with $\bar{A} \subseteq int(\mathcal{M})$, a closed set $B \subseteq A$ and a measurable function $\phi : \mathcal{M} \mapsto B$ with

$$\eta_k = \phi(\pi_{\tau_k}),\ \tau_{k+1} = \inf\{t \geq \tau_k | \pi_t \notin A\} \tag{52}$$

for all $k \in \mathbb{N}_0$.
We denote a CB strategy by (A, B, ϕ). Note that we define ϕ on the domain $\mathcal{M}$ in order to get a well-defined starting value for the new risky fraction process; for the future behaviour of the process only $\phi|_{\partial A}$ is relevant. As the asymptotic return does not depend on the costs for the first transaction, we may assume without loss of generality $\pi_0 \in \partial A$.
Furthermore, in the following let

$$h : \partial A \mapsto \mathbb{R},\ x \mapsto g(x, \phi(x)), \tag{53}$$

where g is the function defined in Remark 2.7.
According to (46), for any CB strategy $K = (A, B, \phi)$

$$R^K(x) = r + \liminf_{t\to\infty} \frac{1}{t} E_x(\sum_{k=0}^{M_t} h(\pi_{\tau_k})). \tag{54}$$

For the rest of this paper we take a CB strategy (A, B, ϕ) as given. According to (54), the asymptotic return does not depend on the starting value π_0, so we may assume without loss of generality $\pi_0 \in \partial A$. The strong Markov property of the price process of the stocks implies that $(\pi_{\tau_k})_{k\in\mathbb{N}_0}$ is (with respect to any measure P_x) a time-homogeneous Markov chain with state space $(\partial A, \mathcal{B}(\partial A))$, where $\mathcal{B}(\partial A)$ denotes the Borel-σ-algebra. Let Q be the transition kernel of $(\pi_{\tau_k})_{k\in\mathbb{N}_0}$. The following theorem, which reveals a technical property of Q that will be very useful in constructing a renewal scheme, is an adaption of Theorem 7.4.3 in [13] to our situation.

3.2 Lemma
Q is a strictly aperiodic Harris kernel with regeneration set ∂A, i.e. there exist a probability measure Ψ on $(\partial A, \mathcal{B}(\partial A))$ and an $\alpha \in (0,1)$ such that for all $x \in \partial A$, $C \in \mathcal{B}(\partial A)$

$$Q(x, C) \geq \alpha \Psi(C). \tag{55}$$

Proof:
Lemma 2.9 ensures that the conditions of Theorem 7.4.3 in [13] are fulfilled. Applying this theorem to our situation yields that for any $x, y \in \partial A$ the measures $Q(x, \cdot)$ and $Q(y, \cdot)$ are equivalent and there exists $\alpha > 0$ with

$$\frac{dQ(x, \cdot)}{dQ(y, \cdot)} \geq \alpha. \tag{56}$$

By choosing an arbitrary $x^* \in \partial A$ and setting $\Psi = Q(x^*, \cdot)$, we get the desired result.

3.3 Remark
The Harris property established above for Q could directly be used to construct a renewal scheme for $(\pi_{\tau_k})_{k\in\mathbb{N}_0}$. But for the the asymptotic return, in addition the times between two trades $\tau_k - \tau_{k-1}$ are of relevance. We thus have to examine the process $(\pi_{\tau_k}, \tau_k - \tau_{k-1})_{k\in\mathbb{N}_0}$ (where we use the convention $\tau_{-1} = 0$). To simplify the notation, we set

$$(Z_k, t_k)_{k\in\mathbb{N}_0} = (\pi_{\tau_k}, \tau_k - \tau_{k-1})_{k\in\mathbb{N}_0}.$$

In the following let α and Ψ be chosen as in Lemma 3.2. To exploit the Harris property of Q, we introduce a transition kernel $\hat{Q}$ by

$$\hat{Q}(x, C) = \frac{Q(x, C) - \alpha\Psi(C)}{1 - \alpha}. \tag{57}$$

Without loss of generality, we may assume that the underlying probability space of our model is rich enough to admit a sequence $(\eta_k)_{k\in\mathbb{N}_0}$ of $\{0,1\}$-valued random variables with the following properties: For all $k \in \mathbb{N}_0$ we have $P(\eta_k = 0) = \alpha$, η_k is independent of $(Z_l, t_l, \eta_{l-1})_{l\leq k}$, and

$$\begin{aligned} & P(Z_{k+1} \in C, t_{k+1} \in D | (Z_l, t_l, \eta_{l-1})_{l\leq k}, \eta_k = 0) \\ = \; & \Psi(C) \int P_{Z_k}(\tau_1 \in D | \pi_{\tau_1} = y) \Psi(dy|C), \qquad (58) \\ & P(Z_{k+1} \in C, t_{k+1} \in D | (Z_l, t_l, \eta_{l-1})_{l\leq k}, \eta_k = 1) \\ = \; & \hat{Q}(Z_k, C) \int P_{Z_k}(\tau_1 \in D | \pi_{\tau_1} = y) \hat{Q}(Z_k, dy|C), \qquad (59) \end{aligned}$$

where these integrals are defined to be 0 if the specified elementary conditional probability measures do not exist.
Hence if $\eta_k = 0$ ("renewal"), Z_{k+1} is generated by Ψ and t_{k+1} is generated by the corresponding conditional distribution. If $\eta_k = 1$, (Z_{k+1}, t_k) is generated in such a way that the unconditional distribution of (Z_{k+1}, t_{k+1}) (prior to the knowledge of η_k) remains unchanged.
In the following, for any $x \in \partial A, t \in [0,\infty), \eta \in \{0,1\}$ let $P_{(x,t,\eta)} = P(\cdot|Z_0 = x, t_k = t, \eta_k = 0)$. For $x \in \partial A$ let $P_x = P(\cdot|Z_0 = x)$ as introduced in (11), and for any probability measure μ on $\mathcal{B}(\partial A)$ let

$$P_\mu = \int P_x(\cdot)\mu(dx). \qquad (60)$$

Furthermore, we consider two different filtrations, which are defined by $\mathcal{A}_k = \sigma((Z_j, t_j)_{0\leq j\leq k})$ and $\mathcal{C}_k = \sigma((Z_j, t_j, \eta_j)_{0\leq j\leq k})$.
Finally, we define random variables $(T_k)_{k\in\mathbb{N}}$ by

$$T_0 = 0, \; T_j = \inf\{k > T_{j-1} : \eta_{k-1} = 0\}. \qquad (61)$$

The following lemma shows, how the Harris property of the Markov chain $(\pi_{\tau_k})_{k\in\mathbb{N}_0}$ can be used to construct a renewal scheme "delayed by one period" for the Markov chain $(Z_k, t_k)_{k\in\mathbb{N}_0}$.

3.4 Lemma
(i) $T_j - 1$ is a stopping time with respect to $(\mathcal{C}_k)_{k\in\mathbb{N}_0}$ for every $j \in \mathbb{N}$, and T_j is a randomized stopping time with respect to $(\mathcal{A}_k)_{k\in\mathbb{N}_0}$ for every $j \in \mathbb{N}_0$, i.e. for all $k \in \mathbb{N}_0$ we have

$$P(T_j > k|\mathcal{A}_k) = P(T_j > k|\mathcal{A}_\infty). \qquad (62)$$

(ii) For all $C \in \mathcal{B}((\partial A \times [0,\infty) \times \{0,1\})^{\mathbb{N}_0})$ and all $j \in \mathbb{N}$

$$P((Z_{T_j+k}, t_{T_j+k}, \eta_{T_j+k})_{k\geq 1} \in C | \mathcal{C}_{T_j-1}) = P_\Psi((Z_k, t_k, \eta_k)_{k\geq 1} \in C). \tag{63}$$

(iii) For all $C \in \mathcal{B}((\partial A \times \{0,1\})^{\mathbb{N}_0})$ and all $j \in \mathbb{N}$

$$P((Z_{T_j+k}, \eta_{T_j+k})_{k\geq 0} \in C | \mathcal{C}_{T_j-1}) = P_\Psi((Z_k, \eta_k)_{k\geq 0} \in C). \tag{64}$$

Proof:
(i) Let $j, k \in \mathbb{N}$. We have

$$\{T_j > k\} = \{|\{m \leq k-1 : \eta_m = 0\}| \leq j-1\} \in \mathcal{C}_{k-1}, \tag{65}$$

thus $T_j - 1$ is a stopping time with respect to $(\mathcal{C}_k)_{k\in\mathbb{N}_0}$.
The second assertion is trivial for $j = 0$ or $k = 0$, so we have to show

$$P(T_j > k | \mathcal{A}_k) = P(T_j > k | \mathcal{A}_\infty) \tag{66}$$

for $j, k \in \mathbb{N}$. Note that for all $C \in \mathcal{B}((\partial A \times [0,\infty) \times \{0,1\})^{\mathbb{N}_0})$

$$\begin{aligned}
&P((Z_{k+l}, t_{k+l}, \eta_{k+l})_{l\geq 1} \in C | \mathcal{C}_{k-1}, \mathcal{A}_k)\\
&= E(P((Z_{k+l}, t_{k+l}, \eta_{k+l})_{l\geq 1} \in C | \mathcal{C}_k) | \mathcal{C}_{k-1}, \mathcal{A}_k)\\
&= E(P_{(Z_k, t_k, \eta_k)}((Y_l, t_l, \eta_l)_{l\geq 1} \in C) | \mathcal{C}_{k-1}, \mathcal{A}_k)\\
&= \alpha P_{(Z_k, t_k, 0)}((Y_l, t_l, \eta_l)_{l\geq 1} \in C) + (1-\alpha) P_{(Z_k, t_k, 1)}((Y_l, t_l, \eta_l)_{l\geq 1} \in C),
\end{aligned}$$

where we used the independence of η_k and $\sigma(\mathcal{C}_{k-1}, \mathcal{A}_k)$ in the last equality. Let

$$\mathcal{C}_{k+1,\infty} = \sigma((Z_{k+l}, t_{k+l}, \eta_{k+l})_{l\geq 1}). \tag{67}$$

The calculation above shows that for all $F \in \mathcal{C}_{k+1,\infty}$

$$P(F | \mathcal{C}_{k-1}, \mathcal{A}_k) = P(F | \mathcal{A}_k), \tag{68}$$

i.e. $\mathcal{C}_{k+1,\infty}$ and $\mathcal{C}_{k-1}$ are independent given $\mathcal{A}_k$. So for all $G \in \mathcal{C}_{k-1}$ we have

$$P(G | \mathcal{A}_k, \mathcal{C}_{k+1,\infty}) = P(G | \mathcal{A}_k), \tag{69}$$

which implies (setting $G = \{T_j > k\}$)

$$P(T_j > k | \mathcal{A}_\infty) = P(T_j > k | \mathcal{A}_k). \tag{70}$$

(ii) Let $C \in \mathcal{B}((\partial A \times [0,\infty) \times \{0,1\})^{\mathbb{N}_0})$ and $j \in \mathbb{N}$. We have

$$\begin{aligned}
& P((Z_{T_j+k}, t_{T_j+k}, \eta_{T_j+k})_{k\geq 1} \in C | \mathcal{C}_{T_j-1}) \\
= \; & E(P((Z_{T_j+k}, t_{T_j+k}, \eta_{T_j+k})_{k\geq 1} \in C | \mathcal{C}_{T_j}) \mathcal{C}_{T_j-1}) \\
= \; & E(P_{(Z_{T_j},0,\eta_{T_j})}((Z_k, t_k, \eta_k)_{k\geq 1} \in C) | \mathcal{C}_{T_j-1}) \\
= \; & E_{(Z_{T_j-1},0,0)}(P_{(Z_1,0,\eta_1)}((Z_k, t_k, \eta_k)_{k\geq 1} \in C)) \\
= \; & \alpha \int P_{(x,0,0)}((Z_k, t_k, \eta_k)_{k\geq 1} \in C) \Psi(dx) \\
& +(1-\alpha) \int P_{(x,0,1)}((Z_k, t_k, \eta_k)_{k\geq 1} \in C) \Psi(dx) \\
= \; & P_\Psi((Z_k, t_k, \eta_k)_{k\geq 1} \in C).
\end{aligned} \tag{71}$$

(iii) This assertion can be proved analogous to (ii).

With respect to P_Ψ, T_1 is geometrically distributed with parameter α, hence $E_\Psi(T_1) = \frac{1}{\alpha} < \infty$. So the unique invariant distribution of $(Z_k)_{k\in\mathbb{N}_0}$ is given by

$$\xi(C) = \alpha E_\psi(\sum_{k=0}^{T_1-1} 1_{\{Z_k \in A\}}) \tag{72}$$

for all $C \in \mathcal{B}(\partial A)$ (see [10], Theorems 10.4.9, 10.4.10). By the usual approximation procedure, we can conclude that for any positive or bounded measurable function $f : \partial A \mapsto \mathbb{R}$

$$\int f(x)\xi(dx) = \alpha E_\Psi(\sum_{k=0}^{T_1-1} f(Z_k)). \tag{73}$$

Part (iii) of the lemma above shows that the future behaviour of the process $(Z_k)_{k\in\mathbb{N}_0}$ starting at any point in time T_j ($j \in \mathbb{N}$) depends neither on j nor on the history of the process. For this reason, the variables $(T_j)_{j\in\mathbb{N}}$ are called regeneration times of $(Z_k)_{k\in\mathbb{N}_0}$.
Lemma 3.4 will enable us in Theorem 3.7 to trace back the asymptotic return to the behaviour of the process $(Z_k, t_k)_{k\in\mathbb{N}_0}$ in a cycle between two regeneration times. The following Lemma 3.5 will then allow us to trace back the asymptotic return to the behaviour in a "typical" period between two trades.

3.5 Lemma
(i) We have

$$\sup_{x\in\partial A} E_x(\sum_{k=0}^{T_2-1} t_k) < \infty \tag{74}$$

and

$$E_\xi(t_1) = \alpha E_\Psi(\sum_{k=T_1}^{T_2-1} t_k). \tag{75}$$

(ii) For all bounded, measurable functions f

$$\sup_{x\in\partial A} E_x(\sum_{k=0}^{T_1-1} |f(Z_k)|) < \infty \tag{76}$$

and

$$E_\xi(f(Z_1)) = \alpha E_\Psi(\sum_{k=0}^{T_1-1} f(Z_k)). \tag{77}$$

Proof:
(i) At first we show

$$\sup_{x\in\partial A} E_x(\tau_1) < \infty. \tag{78}$$

Let $\mathcal{L}$ be the infinitesimal generator of $(\overline{\pi}_t)_{t\in[0,\infty)}$ (see (49)). Furthermore, let $u : \overline{A} \mapsto \mathbb{R}$ be a continuous function which solves $\mathcal{L}u = 1$ on A (for the existence of u see [13], Theorem 3.3.1). By the Feynman-Kac-formula we get for all $x \in \partial A$ and all $m \in \mathbb{N}$

$$E_x(u(\pi_{\tau_1\wedge m})) = u(\phi(x)) + E_x(\int_0^{\tau_1\wedge m} \mathcal{L}u(\pi_s)ds), \tag{79}$$

hence

$$E_x(\tau_1 \wedge m) \leq 2\sup_{y\in\overline{A}} |u(y)|. \tag{80}$$

Taking the limit $m \to \infty$, we obtain the desired result (78) by the monotone convergence theorem.
In the following let

$$M = \frac{\sup_{x\in\partial A} E_x(\tau_1)}{\min\{\alpha, 1-\alpha\}}. \tag{81}$$

For all $x \in \partial A$, $t \in [0,\infty)$ and $D \in \mathcal{B}([0,\infty))$ we have

$$\begin{aligned} P_{(x,t,0)}(t_1 \in D) &= \int P_x(\tau_1 \in D | \pi_{\tau_1} = y)\Psi(dy) \\ &\leq \frac{1}{\alpha}\int P_x(\tau_1 \in D | \pi_{\tau_1} = y)Q(x,dy) \\ &= \frac{1}{\alpha}P_x(\tau_1 \in D), \end{aligned} \tag{82}$$

hence $E_{(x,t,0)}(t_1) \leq M$, and with identical arguments we get $E_{(x,t,1)}(t_1) \leq M$. So we have shown that

$$E_{(x,t,\eta)}(t_1) \leq M \tag{83}$$

for all $(x,t,\eta) \in S$. Now for all $x \in \partial A$ we get

$$\begin{aligned} E_x(\sum_{k=0}^{T_2-1} t_k) &= E_x(\sum_{k=0}^{\infty} t_k 1_{\{T_2>k\}}) \\ &= \sum_{k=0}^{\infty} E_x(E_x(t_k 1_{\{T_2>k\}} | \mathcal{C}_{k-1})) \\ &= \sum_{k=0}^{\infty} E_x(1_{\{T_2>k\}} E_x(t_k | \mathcal{C}_{k-1})) \\ &= \sum_{k=0}^{\infty} E_x(1_{\{T_2>k\}} E_{(Z_{k-1},t_{k-1},\eta_{k-1})}(t_k)) \\ &\leq M\sum_{k=0}^{\infty} P_x(T_2 > k) \\ &= ME_x(T_2) \, < \, \infty, \end{aligned} \tag{84}$$

which proves (74).

We now show equation (75).
Using the fact that $P(T_1 > k|\mathcal{A}_\infty)$ is $\mathcal{A}_k$-measurable, we get

$$\begin{aligned}
E_\xi(t_1) &= \int E_x(t_1)\xi(dx) \;=\; \alpha E_\Psi(\sum_{k=0}^{T_1-1} E_{Z_k}(t_1)) \\
&= \alpha E_\Psi(\sum_{k=0}^{T_1-1} E_\Psi(t_{k+1}|\mathcal{A}_k)) \;=\; \alpha E_\Psi(\sum_{k=0}^{\infty} E_\Psi(t_{k+1}|\mathcal{A}_k)1_{\{T_1>k\}}) \\
&= \alpha \sum_{k=0}^{\infty} E_\Psi(E_\Psi(E_\Psi(t_{k+1}|\mathcal{A}_k)1_{\{T_1>k\}}|\mathcal{A}_\infty)) \\
&= \alpha \sum_{k=0}^{\infty} E_\Psi(P_\Psi(T_1 > k|\mathcal{A}_\infty)E_\Psi(t_{k+1}|\mathcal{A}_k)) \\
&= \alpha \sum_{k=0}^{\infty} E_\Psi(E_\Psi(E_\Psi(t_{k+1}1_{\{T_1>k\}}|\mathcal{A}_\infty)|\mathcal{A}_k)) \\
&= \alpha \sum_{k=0}^{\infty} E_\Psi(t_{k+1}1_{\{T_1>k\}}) \;=\; \alpha E_\Psi(\sum_{k=0}^{T_1-1} t_{k+1}) \\
&= \alpha(E_\Psi(t_{T_1}) + E_\Psi(\sum_{k=1}^{T_1-1} t_k)) \\
&= \alpha(E_\Psi(t_{T_1}) + E_\Psi(E_\Psi(\sum_{k=T_1+1}^{T_2-1} t_k|\mathcal{C}_{T_1-1}))) \\
&= \alpha E_\Psi(\sum_{k=T_1}^{T_2-1} t_k). \qquad (85)
\end{aligned}$$

Here we used Lemma 3.4 (ii) to get the last but one equality.

(ii) Using the boundedness of f, (76) can be shown as (74). Furthermore, we have

$$\begin{aligned}
E_\xi(f(Z_1)) &= \int E_x(f(Z_1))\xi(dx) \;=\; \int\int f(y)Q(x,dy)\xi(dx) \\
&= \int f(x)\xi(dx) \;=\; \alpha E_\Psi(\sum_{k=0}^{T_1-1} f(Z_k)). \qquad (86)
\end{aligned}$$

In principle, we are now in the position to prove the main theorem of this paper, which allows to trace back the asymptotic return of a CB strategy by the behaviour of the risky fractions in a typical period. However, in the proof we will use two renewal theoretic results (Wald's equation, elementary renewal theorem) for r-independent random variables, which are content of the following lemma. Both of them are only slight modifications of those given in [6] and can be proved similarly so we omit the proof.

3.6 Lemma
Let $(X_j)_{j\in\mathbb{N}_0}$ be an integrable stochastic process, adapted to a filtration $(\mathcal{G}_j)_{j\in\mathbb{N}_0}$, such that X_i and X_j are identically distributed for all $i, j \geq 1$ and there exists $r \in \mathbb{N}_0$, such that $(X_{j+i})_{i\geq r+1}$ is independent of $\mathcal{G}_j$ for all $j \in \mathbb{N}_0$. Furthermore, let

$$S_m = \sum_{j=0}^{m} X_j \tag{87}$$

for all $m \in \mathbb{N}_0$.
(i) Let τ be an integrable stopping time (with respect to $(\mathcal{G}_j)_{j\in\mathbb{N}_0}$). Then we have

$$E(S_{\tau+r}) = E(X_0) + E(\tau + r)E(X_1). \tag{88}$$

(ii) Let $X_j \geq 0$ for all $j \in \mathbb{N}_0$, and $E(X_1) > 0$. For all $t \in [0, \infty)$ we set

$$N(t) = \inf\{m \in \mathbb{N} : S_m > t\}. \tag{89}$$

Then

$$\lim_{t\to\infty} \frac{E(N(t))}{t} = \frac{1}{E(X_1)}. \tag{90}$$

3.7 Theorem
For all $x \in \partial A$ we have

$$R^K(x) = r + \frac{E_\xi(h(\pi_{\tau_1}))}{E_\xi(\tau_1)}. \tag{91}$$

Proof:
Let $x \in \partial A$. With M_t defined as in (31), we have

$$M_t = \sup\{k \in \mathbb{N}_0 : \sum_{l=0}^{k} t_l \leq t\}. \tag{92}$$

As h is bounded, from (54) we get

$$R^K(x) = r + \liminf_{t\to\infty} \frac{1}{t} E_x(\sum_{k=0}^{M_t+1} h(Z_k)). \tag{93}$$

For all $j \in \mathbb{N}_0$ let

$$Y_j = \sum_{k=T_j}^{T_{j+1}-1} h(Z_k), \quad d_j = \sum_{k=T_j}^{T_{j+1}-1} t_k, \tag{94}$$

and for all $t \in [0,\infty)$ let

$$N(t) = \inf\{m \in \mathbb{N} : \sum_{j=0}^{m} d_j > t\}. \tag{95}$$

We now show that

$$R^K(x) = r + \liminf_{t\to\infty} \frac{1}{t} E_x(\sum_{j=0}^{N(t)} Y_j). \tag{96}$$

For this, note that $M_t + 1$ is a stopping time with respect to $(\mathcal{C}_k)_{k\in\mathbb{N}_0}$ due to

$$\{M_t + 1 > k\} = \{\sum_{l=1}^{k} t_l \le t\} \in \mathcal{C}_k. \tag{97}$$

Setting $K = \sup_{y\in\partial A} |h(y)|$, the representation (96) now follows from

$$\begin{aligned}
& |E_x(\sum_{j=0}^{N(t)} Y_j) - E_x(\sum_{k=0}^{M_t+1} h(Z_k))| \\
\le\ & E_x(\sum_{k=M_t+2}^{T_{N(t)+1}-1} |h(Z_k)|) \\
\le\ & K E_x((T_{N(t)+1} - 1) - (M_t + 1)) \\
=\ & K E_x(\inf\{k \ge 0 : \eta_{k+M_t+1} = 0\}) \\
=\ & K E_x(E_{(Z_{M_t+1}, t_{M_t+1}, \eta_{M_t+1})}(\inf\{k \ge 0 : \eta_k = 0\})) \\
\le\ & \frac{K}{\alpha},
\end{aligned} \tag{98}$$

For all $m \in \mathbb{N}_0$ we set

$$\mathcal{G}_m = \sigma((Y_j, d_j)_{0\le j\le m}). \tag{99}$$

It can be shown similarly to (97) that $N(t)$ is a stopping time with respect to $(\mathcal{G}_m)_{m\in\mathbb{N}_0}$. According to Lemma 3.4 (iii), $(Y_j)_{j\geq 1}$ is a sequence of P_x-i.i.d random variables. Furthermore, we have $P_x^{Y_1} = P_\Psi^{Y_0}$, and for all $j \geq 1$ Y_j is independent of $\mathcal{G}_{j-1}$. Finally, according to Lemma 3.5 (ii), Y_j is integrable for all $j \in \mathbb{N}_0$. This ensures that we may apply the generalized Wald equation (Lemma 3.6 (i)) to get

$$
\begin{aligned}
R^K(x) &= r + \liminf_{t\to\infty} \frac{1}{t} E_x(\sum_{j=0}^{N(t)} Y_j) \\
&= r + \liminf_{t\to\infty} \frac{1}{t}(E_x(Y_0) + E_x(N(t))E_x(Y_1)) \\
&= r + \liminf_{t\to\infty} \frac{1}{t} E_x(N(t))E_\Psi(Y_0). \qquad (100)
\end{aligned}
$$

Now let

$$
\begin{aligned}
\overline{\mathcal{G}}_m &= \mathcal{G}_{m+1} \text{ für alle } m \in \mathbb{N}_0, \qquad (101)\\
\overline{d}_0 &= d_0 + d_1, \qquad (102)\\
\overline{d}_j &= d_{j+1} \text{ für alle } j \in \mathbb{N}, \qquad (103)\\
S_m &= d_0 + d_1 + \sum_{j=2}^{m+1} d_j \qquad (104)\\
&= \sum_{j=0}^{m} \overline{d}_j \text{ für alle } m \in \mathbb{N}_0. \qquad (105)
\end{aligned}
$$

According to Lemma 3.4 (ii), $(\overline{d}_j)_{j\geq 1}$ is a sequence of P_x-i.i.d. random variables and for all $j \in \mathbb{N}_{\geq 2}$ $\overline{d}_j$ is independent of $\overline{\mathcal{G}}_{j-2}$. Furthermore, according to Lemma 3.5 (i), $\overline{d}_j$ is P_x-integrable for all $j \in \mathbb{N}_0$.
Noting that

$$N(t) + 1 = \inf\{m \in \mathbb{N}_0 : S_m > t\}. \qquad (106)$$

we get by the generalized elementary renewal theorem (Lemma 3.6 (ii)) and Lemma 3.4 (ii)

$$\lim_{t\to\infty} \frac{1}{t} E_x(N(t)+1) = \frac{1}{E_x(\overline{d}_1)} = \frac{1}{E_x(d_2)} = \frac{1}{E_\Psi(d_1)}. \qquad (107)$$

Finally, using Lemma 3.5 we get the desired result

$$
\begin{aligned}
R^K(x) &= r + \liminf_{t\to\infty} E_\Psi(Y_0)\frac{E_x(N(t))}{t} = r + \frac{E_\Psi(Y_0)}{E_\Psi(d_1)} \\
&= r + \frac{E_\xi(h(Z_1))}{E_\xi(t_1)} = r + \frac{E_\xi(h(\pi_{\tau_1}))}{E_\xi(\tau_1)}. \qquad (108)
\end{aligned}
$$

Conclusion

The problem treated in this paper is the generalization of constant boundary strategies from the case of one stock to the case of n stocks and the resulting mathematical treatment. It is shown using methods from diffusion processes, Harris chains, and renewal theory that the asymptotic return is described by the behaviour of the risky fractions in a “typical” period between two trades. This is the content of the foregoing Theorem 3.7 which essentially uses the technical Lemmas 3.2 and 3.4 to 3.6.

Whereas for one stock constant boundary strategies are described by intervals $A = [a, b]$, $B = [\alpha, \beta]$ and choice function $\phi(a) = \alpha$, $\phi(b) = \beta$, we now have in the case of $n > 1$ stocks no restriction on the shape of A, B and ϕ except those from definition 3.1. We conjecture that optimal strategies belong to the class of constant boundary strategies, and it seems to be challenging to prove this and obtain information on the shape of A and B for optimal strategies.

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