Kiel
Working Papers

Kiel Institute
for the World Economy

Chance Theory: A Separation of Riskless and Risky Utility

by

Ulrich Schmidt and Horst Zank

No. 1874 | October 2013

Web: www.ifw-kiel.de

Chance Theory:
A Separation of Riskless and Risky Utility

by Ulrich Schmidt[a] and Horst Zank[b,1]

[a]Department of Economics, Christian-Albrechts-Universität zu Kiel and Kiel Institute for the World Economy, Germany.

[b]Economics, School of Social Sciences, The University of Manchester, United Kingdom.

30 September 2013

Abstract. We present a preference foundation for Chance Theory (CT), a model of decision making under uncertainty where the evaluation of an act depends distinctively on its lowest outcome. This outcome is evaluated with the riskless value function v and the potential increments over it are evaluated by subjective expected utility with a risky utility function u. In contrast to earlier approaches with models that aimed at separating riskless and risky utility, CT does not violate basic rationality principles like first-order stochastic dominance or transitivity. Decision makers with CT-preferences always prefer the expected value of a lottery to the latter, so they are weakly risk averse. Besides explaining behavioral irregularities like the expected utility paradoxes of Allais and Rabin, CT also separates risk attitude in the strong sense from attitude towards wealth. Risk attitude is completely determined by the curvature of u and is independent of the value function v. Conversely, attitude towards wealth is reflected solely through the curvature of v without imposing constraints on u.

Keywords: decision theory, expected utility, riskless utility for wealth, risky utility for money, preference foundation, prudence.

Journal of Economic Literature Classification Numbers: D81.

---

[1]Correspondence: Horst Zank, Economics, School of Social Sciences, The University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom; E-mail: horst.zank@manchester.ac.uk

This paper was written during the second author's visit at the Department of Economics, Christian-Albrechts-Universität zu Kiel, Germany, in summer 2013. Financial support for this visit was provided by the German Academic Exchange Service (DAAD) under the scheme with reference No. A/13/05464.

We received valuable suggestions from Aurélien Baillon. Comments from participants at the seminars at Erasmus University Rotterdam and Cergy Pointoise are also gratefully acknowledged. Remaining errors are ours.

1 Introduction

A specific feature of subjective expected utility (SEU) is that risky and riskless options are evaluated by the same von Neumann-Morgenstern utility function. Already Savage (1954) and Luce and Raiffa (1957) explicitly noted that this function does not necessarily measure strength of preference for certain outcomes as it reflects both attitude towards wealth and risk attitude. Consequently, a concave utility function may merely result from decreasing marginal utility of wealth and need not reflect an intrinsic aversion towards risk (Dyer and Sarin, 1982; Yaari, 1987). Likewise most non-expected utility alternatives to SEU do not distinguish between risky and riskless utility and, thus, cannot separate attitudes towards risk from attitudes towards wealth entirely.

We propose a model, called chance theory (CT), that distinguishes markedly between riskless and risky utility. CT provides a foundation for preferences over acts that depend on their worst outcome evaluated with a value function v and the chances to improve upon that worst outcome. These chances are attached to the possible increments above the worst outcome and are evaluated by SEU with a utility function u defined over those improvements. We call the function v the utility for wealth and the function u the utility for chance. CT generates a separation of attitudes towards wealth and risk attitudes as the latter are completely determined by the curvature of u and independent of v, while the curvature of v reflects only attitudes towards wealth independently of u. Like SEU, CT implies the existence of additive subjective probabilities.[2] Even with additive probabilities and reference independent risk attitudes, CT can accommodate the paradoxes of Allais (1953) and Rabin (2000), and therefore can be regarded as a descriptive alternative to SEU.

Empirical evidence suggests that there may be a fundamental difference between riskless and risky utility. For instance, the common consequence and common ratio effect of Allais (1953) rely on the

[2] Generalizations to non-additive probability measures as in Choquet expected utility (Schmeidler, 1989), in order to allow for ambiguity attitudes (Ellsberg, 1961), are possible in our framework. Likewise, more descriptive generalizations that assume reference-dependence such as in prospect theory (Tversky and Kahneman, 1992), can be derived. As our aim is to focus on a separation of globally consistent risk attitudes and to allow for a direct comparison of CT and SEU, we leave the extension to ambiguity and reference-dependence for future research.

existence of safe options. If these options are moved slightly away from certainty, the violation rate of expected utility (EU) is dramatically reduced (Cohen and Jaffray, 1988; Conlisk, 1989; Andreoni and Sprenger, 2010). In general, EU seems to perform rather well when only risky options over common outcomes are considered (Camerer, 1992; Hey and Orme, 1994; Harless and Camerer, 1994; Starmer, 2000). More direct evidence for a fundamental difference between riskless and risky utility is given by the uncertainty effect (Gneezy et al., 2006; Simonsohn, 2009) where risky options are valued less than their lowest prize is valued under certainty. Also Andreoni and Sprenger (2012) and Andreoni and Harbaugh (2010) find evidence for a disproportionate preference for certainty and propose to use different utility functions for riskless and risky options.

The focus on the worst possible outcome has already been discussed in the theoretical and empirical literature and has been incorporated in various models. Managers disregard investment opportunities that are likely give losses (Payne, et al. 1980, 1981). They are concerned by what in psychology literature is called the security level (Lopes, 1987). Lopes gives the example of farmers, who are particularly concerned with ensuring a certain subsistence level rather than with investing solely in potentially profitable crops. Lopes regards the security motivation as corresponding to weighting the worst outcome in a lottery more heavily than the best outcomes. Moreover, she argued that "sure things have powerful influence in organizing choice" (Lopes 1987, p.278); certainty is special as it allows one to plan, or proceed with plans, without being held back by uncertainty which, by definition, still has to resolve before moving on. The security level plays an important role in several alternatives to expected utility (Jaffray 1988; Gilboa, 1988; Cohen, 1992) but unlike CT these models do not separate riskless and risky utility. Recently, a preference for certainty has been incorporated in the models of Dillenberger (2010) and Cerreia-Vioglio et al. (2013).

The focus on the worst outcome also appears in social welfare analysis in the context of inequality measurement (Rawls, 1971) and in finance (Roy, 1952). Worst outcomes receive special attention also in models of ambiguity aversion (Gilboa and Schmeidler, 1989), and, more recently, in adjustments of

EU by keeping "the best and worst in mind" (Chateauneuf, Eichberger and Grant, 2007; Webb and Zank, 2011). In these latter theories attention is also given to best outcomes (see also Lopes, 1987). Our version of CT does not give special attention to the best outcome, however, the tools presented in this paper can be used to develop extensions of CT that incorporate such potential concerns.

Several axiomatic models that distinguish between riskless and risky utility have been proposed in the literature (Fishburn, 1980; Schmidt, 1998; Bleichrodt and Schmidt, 2002; Diecidue et al., 2004). These models are mainly motivated by the utility of gambling, i.e. an intrinsic utility or disutility of risk (see also Conlisk, 1993; Luce and Marley, 2000). Apart from Bleichrodt and Schmidt (2002) these models assume that safe options are evaluated by a riskless utility whereas risky options are evaluated by the risky utility. If the two utilities are distinct, discontinuities and violations of first-order stochastic dominance are implied. This shows that the purpose of those models is mainly descriptive. However, transparent violations of dominance, as implied by those models, are observed extremely rarely in empirical research. The model of Bleichrodt and Schmidt (2002) implies intransitive preference cycles which are opposite to the cycles reported in the experimental literature (Starmer and Sugden, 1989). Consequently, also that model has drawbacks from a normative and from a descriptive perspective.

CT avoids violations of transitivity and of monotonicity. In fact these properties are explicitly employed. They are complemented (in addition to completeness and continuity) with variants of two familiar principles. The first, a strengthening of the weak cancellation property (Krantz, et al. 1971), ensures that the riskless component of an act (i.e., the worst outcome) is additively separable from its risky remainder and, moreover, that this riskless component is independent of the event that gives the worst outcome. This allows for the interpretation of v as measuring the strength of preference for sure outcomes. The second principle invokes a consistency requirement for potential increments beyond the riskless outcome of an act. Variants of the latter principle have been employed for derivations of subjective expected utility and other generalizations relying on continuous cardinal utility (see Wakker

2010). Here, this consistency requirement allows, as in SEU, to identify a unique probability measure for events and a cardinal measure, u, for improvements beyond the worst outcomes. The latter can then be given the interpretation as capturing risk attitude beyond attitude towards wealth.

The interpretation of outcomes in an act as a sure outcome plus potential improvements has several implications for CT-preferences. These are best seen if one assumes decision under risk with objectively given probabilities. Then a clear separation between attitudes towards wealth and attitudes towards chance is obtained. For instance decreasing absolute risk aversion demands that the function v is concave, while it has no implications for the curvature of the utility of chance u. In that case, a simple application to consumption saving decisions shows that CT-preferences imply prudence, that is, if future income becomes more risky a consumer will save more in the current period (Kimball, 1990). Thus, without knowing the precise shape of the utility for chance one can, from the first and second derivative of the utility of wealth, directly infer whether CT-individuals are precautionary savers, and no higher order derivatives, as in EU, are required. In contrast, demanding aversion to mean-preserving spreads (Rotschild and Stiglitz, 1970) implies that the utility of chance u is concave; no further implication for the utility of wealth v are obtained.

This specific separation into wealth and chance attitudes under CT implies one restriction: irrespective of the curvature of v and of u, at all wealth levels the effect of marginal change in wealth is at least as large as the effect of a corresponding marginal change is on the chance utility. That is, utility of wealth is always at least as steep as the utility of chance. One implication of this relationship between the different utilities is that CT-preferences always display an attraction to the expected value of a lottery, i.e., weak risk aversion holds as standard assumed in economics. This is in agreement with most applications of expected utility theory where a concave utility is typically assumed (e.g., Arrow, 1965, Pratt, 1964) and has a long tradition dating back to Bernoulli (1738, 1954).

The next section introduces our notation and introduces CT formally. Implications of CT for wealth and risk attitudes are presented in Section 3 where also the explanation of the Allais (1953) and the Rabin (2000) paradoxes are reconsidered. Section 4 provides a behavioral foundation for CT. This is followed by a discussion and conclusions in Sections 5 and 6, respectively. All proofs are relegated to the Appendix.

2 Outline of Chance Theory

We consider a finite set of *states*, $\mathcal{S} = \{1, \ldots, n\}$, for a natural number $n \geq 3$, and $\mathcal{A} = 2^{\mathcal{S}}$ is the algebra of subsets of $\mathcal{S}$.[3] Elements $E \in \mathcal{A}$ are called *events*. An *act* f assigns to each state a real valued *outcome*. The set of acts $\mathcal{F}$ can be identified with the Cartesian product space $\mathbb{R}^n$, and hence, we write $f = (f_1, \ldots, f_n)$, where f_s is short for $f(s)$, $s \in \mathcal{S}$.

In CT it is assumed that the decision maker first focuses on the worst outcome when choosing between acts. This is psychologically plausible give the evidence reported in the introduction. Suppose act f has its worst outcome f_m in state m. Then f_m can be regarded as the risk-free wealth of the decision maker when choosing f (or the part of the act that can instantly be integrated into current wealth holdings) and $f_t - f_m, t \neq m$ are the possible improvements in the other states (for which the integration into current wealth is prevented by uncertainty that needs to be resolved). Therefore each act is interpreted as offering a sure outcome and separately a chance for improvements. CT is based on preference conditions such that a decision maker evaluates risk-free wealth with a utility function v and possible increments with a subjective expected utility like evaluation based on a utility function u for chances. Specifically, *Chance Theory* holds if all acts f are evaluated by

$$CT(f) = v(f_m) + \sum_{t \in \mathcal{S}} \pi_t u(f_t - f_m), \qquad (1)$$

where state m is the state with the worst outcome of f, $\pi_t, t \in \mathcal{S}$, are the (positive) *subjective probabilities* of the decision maker generated by a probability measure $\mathcal{P}$ over events, the (*riskless*)

[3] Our results can be extended to infinite state spaces by using tools presented in Wakker (1993b) and Spinu (2012). Identical results for the case of decision under risk, that is, when (objective) probabilities are given, can be derived by applying the procedure of Köbberling and Wakker (2003, Section 5.3).

utility for wealth, $v : \mathbb{R} \to \mathbb{R}$, is strictly increasing and continuous, and the (*risky*) *utility for chance*, $u : \mathbb{R}_+ \to \mathbb{R}_+$, is strictly increasing and continuous with $u(0) = 0$. Further, as we demand that CT is increasing in each outcome, v and u, are related as follows: for all $y \in \mathbb{R}, x > \varepsilon > 0$

$$v(y+\varepsilon) - v(y) > \max_{s \in \mathcal{S}}\{\sum_{t \neq s} \pi_t\}[u(x) - u(x-\varepsilon)]. \tag{2}$$

Under CT the subjective probability measure is uniquely determined and the utility functions are jointly cardinal with location constraints applying to u (i.e., v and u can be replaced by $\tilde{v} = av + b$ and $\tilde{u} = au$ whenever $a > 0$ and $b \in \mathbb{R}$).

It is intuitively plausible that a decision maker focuses on the worst outcome and evaluates possible increments separately and with a different utility function. A similar *segregation* procedure is informally discussed in the editing phase of prospect theory (Kahneman and Tversky, 1979). For acts which have only two strictly positive outcomes (f_1 and f_2), the minimal gain (say f_1) is segregated in prospect theory and and the representation becomes $V(f_1) + \pi_2(V(f_2) - V(f_1))$, where V denotes the prospect theory value function. Our representation differs by the fact that we obtain two separate utility functions v and u and that incremental gains are evaluated directly by u and not by their value difference. Of course a further difference is that in CT the worst consequence is always segregated, even if some outcomes in the considered act are negative. However, outcomes in our framework are interpreted final wealth positions and not in terms of gains and losses relative to a reference point as in prospect theory. Due to limited liability, final wealth should become negative only in exceptional cases.

2.1 Implications for Decision under Risk

For illustration purposes and for further reference it is useful to consider the implication of Inequality (2) for the case of decision under risk. There, the preference $\succcurlyeq$ is defined over *lotteries* $P = (p_1, x_1; \ldots; p_l, x_l)$, i.e., finite probability distributions over monetary outcomes. In that case the lottery $Q = (\delta, y; \frac{1-\delta}{l-1}, y+x; \dots; \frac{1-\delta}{l-1}, y+x)$, with $y \in \mathbb{R}$, $0 < \varepsilon < x$ and $0 < \delta < 1$, is evaluated as

$$CT(Q) = v(y) + (1-\delta)u(x).$$

An improvement of ε in the outcome y that obtains with probability δ gives the lottery $Q' = (\delta, y + \varepsilon; \frac{1-\delta}{l-1}, y+x; \dots; \frac{1-\delta}{l-1}, y+x)$ that first order stochastically dominates Q and has the worst outcome $y + \varepsilon$. The CT-value of Q' is

$$CT(Q') = v(y+\varepsilon) + (1-\delta)u(x-\varepsilon).$$

As CT satisfies strong monotonicity it follows that $CT(Q') > CT(Q)$, or equivalently that

$$v(y+\varepsilon) - v(y) > (1-\delta)[u(x) - u(x-\varepsilon)].$$

The latter inequality holds for any $\delta \in (0,1)$ and, thus,

$$v(y+\varepsilon) - v(y) \geq \sup_{\delta \in (0,1)} \{1-\delta\}[u(x) - u(x-\varepsilon)]$$

follows. We conclude that for decision under risk, the analog condition to Inequality (2) for CT is

$$v(y+\varepsilon) - v(y) \geq u(x) - u(x-\varepsilon) \tag{3}$$

for all $y \in \mathbb{R}, x > \varepsilon > 0$.

Altogether, condition (3) is rather strong as it demands that the minimal slope of v is at least as large as the maximal slope of u. This means that v cannot be too concave or u too convex without violating (3). A similar conclusion can be drawn directly from Inequality (2) if the set of states is sufficiently rich as, e.g., in Savage's (1954) subjective expected utility.

3 Attitudes towards Wealth and Risk

This section mainly assumes CT for decision under risk to ensure comparability with previous studies. We employ the tools of Wakker (1993b) and Köbberling and Wakker (2003) and consider CT for given probabilities and a rich state space. For simplicity and expositional convenience, we also assume that the wealth utility v and the chance utility u are twice continuously differentiable.

3.1 Weak Risk Aversion

It is common to distinguish between weak and strong risk aversion. *Weak risk aversion* holds if an act is always dispreferred to receiving its expected monetary value with certainty. Surprisingly, CT always implies weak risk aversion irrespective of the curvature of u and v. To illustrate this consider the choice between \$50 for sure and a coin flip between zero and \$100. Weak risk aversion demands that $v(50) > v(0) + 0.5u(100)$. Since $u(0) = 0$, the right hand side of this inequality can be reformulated as $v(0)+0.5[u(100)-u(50)]+0.5[u(50)-u(0)]$ so that rearranging yields $v(50)-v(0) > 0.5[u(100) - u(50)] + 0.5[u(50) - u(0)]$. According to condition (3), $v(50) - v(0)$ is larger than both utility differences on the right hand side such that the sure \$50 is valued higher than the lottery. The next proposition shows that this result holds in general.

PROPOSITION 1 *A preference relation which can be represented by CT always displays weak risk aversion.* □

3.2 Strong Risk Aversion

A stronger concept of risk aversion has been proposed by Rothschild and Stiglitz (1970) who define risk aversion as aversion to mean-preserving spreads. For a lottery $P = (p_1, x_1; \ldots; p_l, x_l)$ with strictly positive probabilities $p_1, \ldots, p_l$, a *mean-preserving spread* is given by $P' = (p_1, x_1; \ldots; p_k, x_k + \delta/p_k; \ldots; p_{k'}, x_{k'} - \delta/p_{k'}; \ldots; p_l, x_l)$ whenever $x_k > x_{k'}$ and $\delta > 0$. *Strong risk aversion* demands $P \succ P'$ whenever P' is a mean-preserving spread of P. In EU weak and strong risk aversion are equivalent and both are implied by a concave utility function for wealth. The next result shows, that strong risk aversion in CT is equivalent to having a strictly concave utility function u. This indicates that risk attitude is independent of attitude towards wealth given by the curvature of v.

PROPOSITION 2 *A preference relation which can be represented by CT displays strong risk aversion if and only if u is strictly concave.* □

Although CT-preferences that satisfy stron risk aversion imply that u is strictly concave and also that weak risk aversion holds, the indifference sets of lotteries have convex kinks at the line of certainty. Therefore, they exhibit first-order risk aversion as defined by Segal and Spivak (1990). This means that a CT investor requires a strictly positive premium in order to invest into a risky asset. If the CT-investor is strongly risk averse, he will reduce his investment share if the risk in the asset increases through a zero mean risk on the chance component. This follows from the concavity of u. Clearly, the investor who has invested a share of his wealth into a risky asset will also reduce his share of investment if a zero mean risk added to the asset affects the worst outcome. In that case the statement follows from Inequality (3) rather than from the concavity of u. Subsection 3.5 will discuss investment and savings behavior in a dynamic setting. For that it is important to understand attitudes towards wealth, which are discussed next.

3.3 Attitudes towards Wealth

Although, the curvature of v does not influence whether a subject is chance averse or not, it does determine how weak risk aversion changes when initial wealth increases. For generality we assume decision under uncertainty in this subsection. A subject exhibits *constant* (*decreasing*, respectively, *increasing*) *absolute risk aversion* if $x \sim f$ implies $x + y \sim (\prec, \succ)(f_1 + y, \ldots, f_n + y)$ for all constant acts x, non-constant acts f, and $y > 0$. Expressed in wealth and chance utility terms this means that

$$v(x) = v(f_m) + \sum_{t \in \mathcal{S}} \pi_t u(f_t - f_m) \Rightarrow v(x + y) = (<, >) v(f_m + y) + \sum_{t \in \mathcal{S}} \pi_t u(f_t - f_m) \quad (4)$$

where $f_m = \min_{s \in \mathcal{S}}\{f_s\}$, which yields

$$v(x+y) - v(x) = (<,>) v(f_m + y) - v(f_m). \tag{5}$$

This immediately implies the following result, since by monotonicity $x > f_m$.

PROPOSITION 3 *A Chance Theory preference relation displays constant (decreasing, increasing) absolute risk aversion if and only if v is linear (concave, convex).* □

Having seen how risk attitude is decomposed into sensitivity to absolute changes in wealth (captured by v) and sensitivity to chance (captured by u) the next subsection looks at comparative risk attitudes.

3.4 Comparative Risk Attitudes

An individual A is defined to be more risk averse than another individual B if A always dislikes mean-preserving spreads to a greater extent than B does. In EU this is equivalent to A having for all lotteries a lower certainty equivalent than B. In CT matters are somewhat more complex as we have to distinguish whether a mean-preserving spread does change the worst outcome or not. In the latter case only u does influence the comparative risk aversion between A and B whereas in the former case both u and v are involved.

As a preparation we formally recall the classical definition of a certainty equivalent of a lottery, which we supplement with the analog notion of a (conditional) chance equivalent. For a lottery $P = (p_1, x_1; \cdots; p_l, x_l)$ and a preference relation $\succcurlyeq$, the *certainty equivalent*, $CE(P)$, is defined as the outcome which, when obtained with certainty, is indifferent to the lottery P, i.e., $(1, CE(P)) \sim P$. Similarly, for $x_m \leq x_i, i = 1, \ldots, l$ we define the *conditional certainty equivalent* $CCE(P)$ as the outcome which, when obtained with probability $1 - \mu$ or otherwise obtaining x_m, is indifferent to the lottery that gives P with probability $1 - \mu$ or otherwise x_m for all $\mu \in (0, 1)$, i.e., $(\mu, x_m; 1 -$

$\mu, CCE(P)) \sim (\mu, x_m; 1-\mu, P)$ for any $\mu \in (0,1)$. For CT-preferences the CCE is independent of the value $\mu \in (0,1)$. This allows us to define comparative risk aversion of $\succsim_A$ and $\succsim_B$ as follows: A is *more risk averse* than B if we have $CE_A(P) < CE_B(P)$ and $CCE_A(P) < CCE_B(P)$ for all lotteries P.

PROPOSITION 4 *Consider two preference relations $\succsim_A$ and $\succsim_B$ which can be represented by CT. Then A is more risk averse than B if and only if $u''_A(x)/u'_A(x) > u''_B(x)/u'_B(x)$ and $v'_A(y)/u_A(x) > v'_B(y)/u_B(x)$ for all $y, x > 0, y \in \mathbb{R}$.* □

Proposition 4 indicates that in order to compare risk attitudes of two decision makers we require a comparison of the absolute degree of curvature of their chance utilities and, further, a comparison of the changes in the wealth utility relative to chance utility. Clearly, if both decision makers share the same utility for wealth, then A is more risk averse than B if and only if $u''_A(x)/u'_A(x) > u''_B(x)/u'_B(x)$. This implies that u_A is a concave transformation of u_B, as required by first inequality in Proposition 4. Conversely, if both decision makers share the same chance utility, then the second inequality in Proposition 4 reduces to $v'_A(y) > v'_B(y)$. This means that individual A appreciates changes in wealth more that individual B does. This, together with Inequality (3) implies that A displays more risk aversion than B does. In general, however, a comparison of risk behavior requires a comparison of tradeoffs for chance and a comparison of tradeoffs between wealth and chance for both individuals.

3.5 Prudence

According to Kimball (1990) an agent is prudent if adding an insurable zero-mean risk to his future wealth raises his optimal saving. This precautionary saving can be analyzed in a simple model with two periods, 0 and 1 and exponential discounted utility. Let w_0 be the deterministic wealth in period 0 whereas wealth in period 1 is given by the random variable $\tilde{w}_1$. With expected utility theory preferences, optimal savings S are determined by maximizing

$$U(w_0 - S) + \beta E(U(\tilde{w}_1 + \rho S)), \tag{6}$$

where β is the time discount factor and $\rho = 1 + r$ for the interest rate r, E is the expectation operator and U is a standard von-Neumann-Morgenstern utility. If $\underline{w}_1$ denotes the minimum wealth in period 1, the equivalent expression under CT is

$$v(w_0 - S) + \beta[v(\underline{w}_1 + \rho S) + E(u(\tilde{w}_1 - \underline{w}_1))], \tag{7}$$

and the first order condition for optimal savings S^* becomes

$$v'(w_0 - S^*) = \beta\rho v'(\underline{w}_1 + \rho S^*) \tag{8}$$

If the riskiness of $\tilde{w}_1$ increases without changing $\underline{w}_1$, optimal savings remain unchanged. Then the individual can continue with his "habitual" or planned consumption of at least $\underline{w}_1$ tomorrow. If, however, increasing risk leads to a decrease of $\underline{w}_1$, optimal savings increase, if the curvature of v does not switch from concave to convex or vice versa, i.e. v is either globally concave, globally convex or linear. Obviously, for small changes in $\underline{w}_1$ the requirement that the sign of v'' at $w_0 - S^*$ and at $\underline{w}_1 + \rho S^*$ is the same will also suffice for locally prudent behavior. Remarkably, whatever the risk attitudes captured by the utility of chance, they do not have any influence on the level of savings. We summarize the result as follows:

Proposition 5 *An agent with Chance Theory preferences is prudent if the agent exhibits constant, decreasing or increasing absolute risk aversion.* □

3.6 Paradoxes

This section shows that CT is able to resolve several behavioral irregularities which are inconsistent with EU and have motivated the development of alternative descriptive theories like prospect theory. The most prominent evidence against EU are the common consequence and common ratio effects of Allais (1953). An example for the common consequence effect is given by the following two lottery pairs:

A:	100% chance of \$1 million	*versus*	B:	10% chance of \$5 million 89% chance of \$1 million 1% chance of \$0
A*:	11% chance of \$1 million 89% chance of \$0	*versus*	B*:	10% chance of \$5 million 90% chance of \$0

In this example many people choose A in the first choice problem and B* in the second one whereas EU demands that subjects either choose A and A* or B and B*. The typical preferences in terms of CT yield $v(1) > v(0) + 0.89u(1) + 0.1u(10)$ for the first pair and $v(0) + 0.11u(1) < v(0) + 0.1u(5)$ for the second one. It is obvious that both inequalities can be satisfied simultaneously. Subtracting the second inequality from the first one yields $v(1) - v(0) - 0.11u(1) > 0.89u(1)$ which implies $v(1) - v(0) > u(1) - u(0)$ since $u(0) = 0$. This latter inequality is always satisfied in CT due to monotonicity, see Inequality (3). Hence, CT is either consistent with the implications under EU or it implies a violation in the typical direction. A violation in the opposite direction is excluded by monotonicity under CT.

An example for the common ratio effect is given by the following two lottery pairs:

C:	100% chance of \$3000	*versus*	D:	80% chance of \$4000 20% chance of \$0
C*:	25% chance of \$3000 75% chance of \$0	*versus*	D*:	20% chance of \$4000 80% chance of \$0

EU demands that subjects choose either C and C* or D and D* whereas there is evidence that many people prefer C and D*. For this choice pattern CT yields $v(3000) > v(0) + 0.8u(4000)$ and

$v(0) + 0.25u(3000) < v(0) + 0.2u(4000)$. Rearranging the second inequality to $u(3000) < 0.8u(4000)$ and subtracting it from the first one, implies $v(3000) - v(0) > u(3000) - u(0)$ which holds due to monotonicity. Therefore, also for the common ratio effect CT is either consistent with EU or a violation of EU in the observed direction whereas violations in the opposite direction are excluded.

A further paradox challenging the descriptive validity of EU has been put forward by Rabin (2000). Rabin argues that many people would reject a coin flip where they either win \$11 or loose \$10 and they would do so at all initial wealth levels. Rabin's calibration theorem shows that this small-stake risk aversion implies unrealistic high degrees of risk aversion for larger stakes. In CT, rejecting the coin flip at initial wealth w implies $v(w) > v(w-10) + 0.5u(21)$. From condition (3) we know that $v(w) - v(w-10) > u(10)$. Therefore, a sufficient condition for resolving the Rabin paradox is $u(10) > 0.5u(21)$, e.g., a slight concavity of u at small stakes without any implications for behavior at higher stakes.

4 Preference Foundation

To simplify the exposition, we use $f_E g$ for an act that agrees with the act f on event E and with the act g on the complement E^c. Also, we use $h_s f$ instead of $h_{\{s\}} f$ for any state $s \in \mathcal{S}$. Sometimes we identify constant acts with the corresponding outcome. We may thus write $f_E x$ for an act agreeing with f on E and giving outcome x for states $s \in E^c$. Similarly, we write $x_E f$ for an act agreeing with f on E^c and giving outcome x for states $s \in E$.

We consider a preference relation $\succcurlyeq$ on the set of acts. As usually, $f \succcurlyeq g$ means that act f is weakly preferred to act g. The symbols $\succ$ and $\sim$ denote strict preference and indifference, respectively. The preference relation $\succcurlyeq$ is a *weak order* if it is *complete* ($f \succcurlyeq g$ or $g \succcurlyeq f$ for any acts f, g) and transitive. A functional $V : \mathcal{F} \to \mathbb{R}$ *represents* the preference relation $\succcurlyeq$ if for all $f, g \in \mathcal{F}$ we have $f \succcurlyeq g \Leftrightarrow V(f) \geq V(g)$.

We recall some standard properties for the preference $\succcurlyeq$ and the corresponding result for a representation V. The preference relation $\succcurlyeq$ on $\mathcal{F}$ satisfies *monotonicity* if $f \succ g$ whenever $f_s \geq g_s$ for all states $s \in \mathcal{S}$ with a strict inequality for at least one state. By employing this condition we ensure that the subjective probabilities, derived later, are positive. This follows because monotonicity excludes null states, that is, states where the preference is independent of the magnitude of outcomes. Formally, a state s is *null* if $x_s f \sim y_s f$ for all acts f and all outcomes x, y.

The continuity condition defined here is continuity with respect to the Euclidean topology on $\mathbb{R}^n$: $\succcurlyeq$ satisfies *continuity* if for any act f the sets $\{g \in \mathcal{F} | g \succcurlyeq f\}$ and $\{g \in \mathcal{F} | g \preccurlyeq f\}$ are closed subsets of $\mathbb{R}^n$. Following Debreu (1954) we know that the preference relation $\succcurlyeq$ is a continuous monotonic weak order if and only if there exists a representation $V : \mathcal{F} \to \mathbb{R}$ for $\succcurlyeq$ that is continuous and strictly increasing in each argument. Further, this representation is continuously ordinal, that is, V is unique up to strictly increasing continuous transformations.

The representation of CT is, besides the standard assumptions of weak ordering, continuity, and monotonicity, obtained by imposing two further conditions. The first ensures independence of risk-free wealth from possible increments and the second one demands independence between the possible increments. Although the motivation for proposing CT is mainly descriptive, we see no obvious reasons to dismiss these conditions from a normative viewpoint. Besides transitivity, monotonicity seems to be the most important requirement for this viewpoint.

Our goal is to restrict the general representation V by requiring a form of additive separability, more restrictive than traditional additive separability (Debreu 1960). To this aim we require further notation. By $\mathcal{F}_m := \{f \in \mathcal{F} | f_t \geq f_m, t = 1, \ldots, n\}$ we denote the set of acts that have the worst possible outcome in state $m \in \mathcal{S}$. To simplify the presentation of the next preference conditions we write $(m : \alpha; g) := \alpha_m(g + \alpha)$ for an act which has its minimal outcome α in state m and the outcome $g_s + \alpha$ in all other states $s \neq m$.

Next we assume a strengthening of the triple cancellation property which has already been used to obtain additive separability (e.g., Wakker 1989).[4] The preference relation $\succcurlyeq$ satisfies *consistent worst*

[4] Traditionally, triple cancellation has been formulated with weak preferences. Given the structural assumption is our

outcome segregation (WOS) if for all $m, m' \in \mathcal{S}$:

$$(m:\alpha;f)\sim(m:\beta;g) \quad , \quad (m:\gamma;f)\sim(m:\delta;g)$$
$$\text{and } (m':\alpha;f')\sim(m':\beta;g') \quad \Rightarrow \quad (m':\gamma;f')\sim(m':\delta;g').$$

WOS entails three important requirements. First, it demands a segregation of the worst outcome from all other possible outcomes. Special attention is given to the worst outcome and separately to changes from that worst outcome within an act. Secondly, when $m = m'$, the condition demands a condition akin of triple cancellation. The latter has frequently been used in the derivation of additive separability when the objects of choice are two dimensional. In our case one can interpret the first dimension as the state m and the second dimension as the residual states. The third requirement of WOS is a consistency requirement for equivalent worst outcome tradeoffs. The first two indifferences, $(m:\alpha;f)\sim(m:\beta;g)$ and $(m:\gamma;f)\sim(m:\delta;g)$, indicate that obtaining α instead of β as worst outcomes is equivalent to obtaining γ instead of δ as worst outcomes when the preference difference for the potential improvements in f and g that can be obtained in states other than m are neutralized. Similarly, the latter two indifferences in WOS indicate that trading off the worst outcome α for β is equivalent to trading off the worst outcome γ for δ given the preference difference for the potential improvements in f' and g' that can be obtained states other than m'. The consistency requirement says that such equivalent worst outcome tradeoffs are independent of the states where they are observed. This consistency requirement is similar in spirit to the tradeoff consistency property recently proposed by Alon (2013), where consistency of tradeoffs is required for worst outcomes but also, separately, for best outcomes. The difference here is that we focus exclusively on the worst outcomes and, additionally, equivalent tradeoffs are measured conditional on improvements from the worst outcomes. Combining WOS with the rationality principles of weak order and monotonicity, and requiring continuity we

paper and the properties of monotonicity and continuity, it will suffice for our purposes to formulate the next property using indifferences only. See also Köbberling and Wakker (2003, p. 409) who argue for tradeoff consistency, a different extension of triple cancellation, formulated with indifferences instead of preferences because of additional transparency due to symmetry of the indifference relation.

obtain the following result.

Lemma 6 *Assume that the preference relation $\succcurlyeq$ on $\mathcal{F}$ is a continuous monotonic weak order that satisfies consistent worst outcome segregation. Then, there exists continuous strongly monotonic functions $v : \mathbb{R} \to \mathbb{R}, U_{-s} : \mathbb{R}_+^{n-s} \times \{0\} \times \to \mathbb{R}_+^{s-1}$ with $U_{-s}(0, \ldots, 0) = 0, s = 1, \ldots, n$, such that on each set $\mathcal{F}_s$ the preference $\succcurlyeq$ is represented by*

$$W(f) = v(f_s) + U_{-s}(f - f_s).$$

Further, for $a > 0$ and real-valued b, the function v can be replaced by $av + b$ whenever U_{-s} is replaced by aU_{-s}, $s \in \{1, \ldots, n\}$. □

Next we are concerned with preference conditions that allow for additive separability of the function $U_{-s}, s = 1, \ldots, n$. In fact our conditions are stronger because they require proportionality of the resulting additively decomposed functions, which, in turn, allows for the identification of subjective probabilities for states.

We require that the preference relation $\succcurlyeq$ satisfies *tradeoff consistency for chance*:

$$\begin{aligned} \text{if } (f_m + w)_t f &\sim (f_m + x)_t g \text{ and } (f_m + y)_t f \sim (f_m + z)_t g \\ \text{then } (f'_m + w)_{t'} f' &\sim (f'_m + x)_{t'} g' \text{ implies } (f'_m + y)_{t'} f' \sim (f'_m + z)_{t'} g', \end{aligned}$$

whenever all acts are from the same set $\mathcal{F}_m, m = 1, \ldots, n$ and $t, t' \neq m$.

Note that the principle of tradeoff consistency for chance is meaningless if there are exactly two states of nature.

Theorem 7 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on $\mathcal{F}$.*

(i) The preference relation $\succcurlyeq$ is represented by CT, with (positive) subjective probabilities $\pi_s, s = 1, \ldots, n$, and strictly increasing continuous functions $v : \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{R}_+ \to \mathbb{R}_+$, with $u(0) = 0$

such that for all $y \in \mathbb{R}, x > \varepsilon > 0$ Eq. (2) holds.

(ii) *The preference relation satisfies weak order, monotonicity, continuity, worst outcome separability and tradeoff consistency for chance.*

Further, the subjective probabilities are uniquely determined and v, u can be replaced by $\tilde{v} = av + b$ and $\tilde{u} = au$ whenever $a > 0$ and $b \in \mathbb{R}$. □

5 Discussion

The separation of overall utility into a risk free utility for wealth and a risky utility for chance in CT implies very specific risk behavior. Sure improvements in the wealth of an agent are more valuable than a risky distribution of improvements which are of equal expectation. As this classical notion of weak risk aversion is satisfied by all CT-agents, the latter will only invest in risky assets if the expected return on such assets exceeds a certain positive amount. Therefore CT-preferences can explain the equity premium puzzle and certainty effects. While it is customary in economic application to assume that agents are risk averse, such behavior is restrictive as there are situations in which risk seeking is observed. Since at least Friedman and Savage (1948) it has been clear that simultaneous gambling and purchase of insurance calls for an explanation. In the simplest form presented here, CT cannot explain this phenomenon. One might be tempted to invoke as extension of CT a non-additive measure for events instead of a probability measure. For decision under risk this means that decision weights as in rank-dependent utility (Quiggin, 1981; 1982) or Choquet expected utility (Schmeidler, 1989) and modern prospect theory (Tversky and Kahneman, 1992) are obtained. In those theories, decision weights do indeed offer an explanation for the simultaneous purchase of lottery tickets and insurance (Wakker 2010, p.207). For CT, while possible, such extensions will not be able to serve as explanation for this phenomenon as they will also call for an analogous condition to Inequality (2) with decision weights replacing probabilities. The new condition will require that the decision weights cannot be too large in addition to the utility for chance having to be everywhere flatter than the riskless utility at all wealth levels. Thus, probability weighting functions cannot be too steep either, which prevents risk seeking behavior. We conjecture that, in general, an account for a preference or utility for gambling is prevented due to our joint requirement of monotonicity and the particular separation of outcomes in an act into a riskless and a chance component. As discussed in Diecidue, et al. (2004) theories accounting for the utility of gambling typically violate monotonicity or transitivity. These are basic rationality principles that we like to preserve here. However, the replacement of subjective probabilities by decision weights will enable CT to accommodate the paradoxes of Ellsberg (1961) and therefore enrich CT with the possibility to study ambiguity attitudes, which has been a very popular research theme in recent decades.

To explain the simultaneous purchase of insurance and lottery tickets or, more generally, to allow for risk seeking behavior, a different approach is required. It is the special role of the worst outcome that needs to be changed. A simple extension would constitute of some weighted average between the CT-value of an act and the dual analog of CT, where the maximum of an act is given the special role, e.g., resulting in a Hurwicz-like criterion for the evaluation function (see Arrow and Hurwicz, 1972). Alternatively, one can envisage that any outcome of an act ranging between the worst and the best outcome is designated as special. This is different to the assumption of a target or reference point that is fixed in advance (Fishburn, 1977; Kahneman and Tversky, 1979) as these are independent of the acts that are considered. Such an approach has been provided in Gul's (1991) disappointment aversion model, where the certainty equivalent, CE, of a lottery is designated as a special outcome. There a lottery is viewed as an α-probability mixture of two lotteries, the first having prizes below the CE and the second having prizes above the CE. The overall value of a lottery is then given by a weighted average of the expected utility of these two lotteries where the weights are a function of α and a parameter β that measures the degree of disappointment. As in expected utility, Gul's model does not specify if outcomes are final wealth positions or changes in wealth. We think that a generalization of CT, with the CE of an act designated as having a prominent role or value in the evaluation of acts, may give a useful extension. Such an extension would feature three utility functions: one for wealth, that is assigning real values to the CE of an act; a chance utility for improvements, assigning positive value to changes in outcomes above the CE; and its counterpart, a chance utility for decrements, assigning negative value to the latter. Such a model can also provide an extension to prospect theory (Kahneman and Tversky, 1979), with the chance utility interpreted as prospect theory's value function and the utility for wealth interpreted as valuing the lottery-dependent reference point; its derivation, however, lies beyond the scope of this paper.

6 Conclusion

In order to achieve a separation of attitudes towards wealth and attitudes towards risk we have proposed the interpretation of acts as a composition of the worst possible outcome plus likely improvements, each valued by separate utility functions. Chance theory is a descriptive theory that deviates significantly from the expected utility model. It agrees with the latter only for risk neutral preferences. Remarkably, CT does not need higher order moments to explain prudence and risk aversion, and suggests that these behaviors may be unrelated. Our model can be a reasonable approach for the analysis of the most plausible type of behavior. If people are not put in situations that they dislike and which, whenever possible, they avoid, or if they admit that the dream of winning a fortune in the lottery will remain a life-long dream for the majority of them, then most likely people are risk averters. They will value sure things more than risky ones and reveal risk averse behavior, as does the classical economic agent. For such agents we have developed Chance Theory.

Appendix: Proofs

Proof of Proposition 1: We have to show that for all lotteries $P = (p_1, x_1; \ldots; p_l, x_l)$ obtaining the expected value of the lottery, $EV(P)$, for sure is weakly preferred to the lottery P. That is,

$(1, EV(P)) \succcurlyeq P$ for all lotteries P.

Let x_m be the smallest outcome of some arbitrary lottery P. Substitution of CT into the previous preference gives

$$v(EV(P)) \geq v(x_m) + \sum_{i=1}^{l} p_i u(x_i - x_m),$$

which is equivalent to

$$v(EV(P)) - v(x_m) \geq \sum_{i=1}^{l} p_i u(x_i - x_m). \tag{9}$$

Observe that for any $k \in \mathbb{N}$ the following identity holds

$$v(EV(P)) - v(x_m) = \sum_{j=1}^{k} \{v(EV(P) - \frac{j-1}{k}[EV(P) - x_m]) - v(EV(P) - \frac{j}{k}[EV(P) - x_m])\}.$$

By setting

$$\Delta v_k^+ := \min_{j \in \{1,\dots,k\}} \{\frac{v(EV(P) - \frac{j-1}{k}[EV(P) - x_m]) - v(EV(P) - \frac{j}{k}[EV(P) - x_m])}{\frac{1}{k}[EV(P) - x_m]}\}$$

we obtain the following inequality

$$\begin{aligned} v(EV(P)) - v(x_m) &\geq \sum_{j=1}^{k} \frac{\Delta v_k^+}{k}[EV(P) - x_m] \\ &= \Delta v_k^+ [EV(P) - x_m]. \end{aligned}$$

Let $y \in [x_m, EV(P)]$ be an outcome where the slope of v is smallest. Such an outcome exists as we assumed that v is continuously differentiable over $\mathbb{R}$. Then, there exists $\delta > 0$ such that $\Delta v_k^+ \geq [v(y + \varepsilon) - v(y)]/\varepsilon$ for all $0 < \varepsilon < \delta$. Note that the latter inequality is strict, unless v is linear on $[x_m, EV(P)]$. It then follows that

$$v(EV(P)) - v(x_m) \geq \frac{v(y + \varepsilon) - v(y)}{\varepsilon}[EV(P) - x_m] \tag{10}$$

for all $0 < \varepsilon < \delta$ with a strict inequality unless v is linear on $[x_m, EV(P)]$.

Next, observe that for any $k_i \in \mathbb{N}$, $i = 1, \ldots, l$, the following identity holds

$$\sum_{i=1}^{l} p_i u(x_i - x_m) = \sum_{i=1}^{l} p_i \sum_{j_i=1}^{k_i} \{u(x_i - \frac{j_i - 1}{k_i}(x_i - x_m)) - u(x_i - \frac{j_i}{k_i}(x_i - x_m))\}.$$

By setting for each $i = 1, \ldots, l$

$$\Delta u_{k_i}^{-} := \max_{j_i \in \{1,\ldots,k_i\}} \{\frac{u(x_i - \frac{j_i-1}{k_i}(x_i - x_m)) - u(x_i - \frac{j_i}{k_i}(x_i - x_m))}{\frac{1}{k_i}[x_i - x_m]}\}$$

we obtain the inequality

$$\sum_{i=1}^{l} p_i u(x_i - x_m) \leq \sum_{i=1}^{l} p_i \Delta u_{k_i}^{-}(x_i - x_m).$$

Further, by setting

$$\Delta u_P^{-} := \max_{i \in \{1,\ldots,m\}} \{\Delta u_{k_i}^{-}\}$$

we obtain

$$\begin{aligned} \sum_{i=1}^{l} p_i u(x_i - x_m) &\leq \Delta u_P^{-} \sum_{i=1}^{l} p_i (x_i - x_m) \\ &= \Delta u_P^{-}[EV(P) - x_m]. \end{aligned}$$

Let $x \in [0, \max_{i \in \{1,\ldots,l\}}\{x_i - x_m\}]$ be an outcome where the slope of u is largest.[5] Such an outcome exists as u is continuously differentiable. Then, there exists $\delta' > 0$ such that $\Delta u_P^{-} \leq [u(x) - u(x - \varepsilon)]/\varepsilon$ for all $0 < \varepsilon < \delta'$, with a strict inequality unless u is linear on $[0, \max_{i \in \{1,\ldots,l\}}\{x_i - x_m\}]$. It then follows that

$$\sum_{i=1}^{l} p_i u(x_i - x_m) \leq \frac{u(x) - u(x - \varepsilon)}{\varepsilon}[EV(P) - x_m] \tag{11}$$

for all $0 < \varepsilon < \delta'$ with a strict inequality unless u is linear on $[0, \max_{i \in \{1,\ldots,l\}}\{x_i - x_m\}]$.

[5] If $x = 0$ take $x^* > 0$ sufficiently close to 0.

Let $\delta^* = \min\{\delta, \delta'\}$. Then, the Inequalities (10) and (11) above hold for all $0 < \varepsilon < \delta^*$. Recall that Inequality (3) in the main text also holds, i.e.,

$$v(y' + \varepsilon') - v(y') \geq u(x') - u(x' - \varepsilon')$$

for all $y' \in \mathbb{R}, x' > \varepsilon' > 0$, and thus, it holds in particular for $y' = y, x' = x$ and $\varepsilon' = \varepsilon$ for all $0 < \varepsilon < \delta^*$. Therefore,

$$\frac{v(y + \varepsilon) - v(y)}{\varepsilon}[EV(P) - x_m] \geq \frac{u(x) - u(x - \varepsilon)}{\varepsilon}[EV(P) - x_m]$$

holds for all $0 < \varepsilon < \delta^*$. Invoking Inequality (10) for the term on the left and Inequality (11) for the term on the right implies Inequality (9) as desired. Further, Inequality (9) is strict unless u is linear on $[0, \max_{i \in \{1,\ldots,l\}}\{x_i - x_m\}]$ or v is linear on $[x_m, EV(P)]$.

As P was an arbitrary non-degenerate lottery, it follows that CT implies weak risk aversion for decision under risk. This completes the proof of Proposition 1. □

Proof of Proposition 2: We have to show that strong concavity of u under CT is equivalent to $P = (p_1, x_1; \ldots; p_l, x_l) \succ P' = (p_1, x_1; \ldots; p_k, x_k + \delta/p_k; \ldots; p_{k'}, x_{k'} - \delta/p_{k'}; \ldots; p_l, x_l)$ whenever $x_k > x_{k'}$ and $\delta > 0$.

Step (i): Suppose that $x_{k'} - \delta/p_{k'}$ is not the minimal outcome of P', i.e. P and P' have the same minimal outcome. In this case CT reduces to an expected utility representation. As a result, it follows that u is concave. Conversely, strict concavity of u is sufficient for $P \succ P'$ if $x_{k'} - \delta/p_{k'}$ is not the minimal outcome of P'.

Step (ii): Suppose now that $x_{k'}$ is the minimal outcome of P such that $x_{k'} - \delta/p_{k'}$ is the minimal outcome of P'. We have to show that $P \succ P'$ or, equivalently, after substitution of CT that

$$\begin{aligned} v(x_{k'}) + \sum_{i=1}^{l} p_i u(x_i - x_{k'}) \quad > \quad & v(x_{k'} - \delta/p_{k'}) + \sum_{\substack{i=1 \\ i \neq k,k'}}^{l} p_i u(x_i - x_{k'} + \delta/p_{k'}) + \\ & p_k u(x_k + \delta/p_k - x_{k'} + \delta/p_{k'}) + p_{k'} u(0). \end{aligned} \tag{12}$$

As u is strictly concave it holds that

$$p_k u(x_k + \delta/p_k - x_{k'} + \delta/p_{k'}) + p_{k'} u(0) < p_k u(x_k - x_{k'} + \delta/p_k) + p_{k'} u(\delta/p_{k'}).$$

Therefore it remains to show that

$$v(x_{k'}) + \sum_{i=1}^{l} p_i u(x_i - x_{k'}) > v(x_{k'} - \delta/p_{k'}) + \sum_{i=1}^{l} p_i u(x_i - x_{k'} + \delta/p_{k'}). \tag{13}$$

Rearranging yields

$$\begin{aligned} v(x_{k'}) - v(x_{k'} - \delta/p_{k'}) \quad & > \quad \sum_{i=1}^{l} p_i [u(x_i - x_{k'} + \delta/p_{k'}) - u(x_i - x_{k'})]. \\ & \Leftrightarrow \\ \sum_{i=1}^{l} p_i [v(x_{k'}) - v(x_{k'} - \delta/p_{k'})] \quad & > \quad \sum_{i=1}^{l} p_i [u(x_i - x_{k'} + \delta/p_{k'}) - u(x_i - x_{k'})] \end{aligned} \tag{14}$$

From condition (3) we know that the utility of wealth difference on the left side of the exceeds each chance utility difference on the right hand side of the inequality. Therefore, $P \succ P'$ follows whenever $x_{k'}$ is the minimal outcome of P.

Step (iii): It remains to consider the case that $x_{k'}$ is not the minimal outcome of P but $x_{k'} - \delta/p_{k'}$ is the minimal outcome of P'. Suppose that x_m is the minimal outcome of P. We have $x_{k'} > x_m \geq x_{k'} - \delta/p_{k'}$. If $x_m = x_{k'} - \delta/p_{k'}$, then $P \succ P'$ follows from strict concavity of u using arguments similar to those in Step (i). Alternatively, if $x_{k'} > x_m > x_{k'} - \delta/p_{k'}$, then we can the mean-preserving spread from P to P' can be obtained from two successive mean-preserving spreads follows. Define δ' such that $x_m = x_{k'} + \delta'/p_{k'}$ and set $P'' = (p_1, x_1; \ldots; p_k, x_k + \delta/p_k; \ldots; p_{k'}, x_{k'} - \delta/p_{k'}; \ldots; p_l, x_l)$. By construction P'' is a mean-preserving spread of P and both lotteries have the same minimal outcome. Thus, $P \succ P''$ follows as u is concave. Further, P' is a mean-preserving spread of P'' with minimal outcome $x_{k'} + \delta'/p_{k'}$ and $x_{k'} + \delta'/p_{k'} - \delta/p_{k'}$. By Step (ii) it then follows that $P'' \succ P'$. Thus, $P \succ P''$, $P'' \succ P'$ and transitivity yields implies $P \succ P'$. Thus, Steps (i)–(iii), complete the proof of Proposition 2. □

Proof of Proposition 3: The proof follows immediately from the arguments provided preceding the proposition in the main text. □

Proof of Proposition ??: Let $P = (p_1, x_1; \ldots; p_l, x_l)$ be a non-degenerate lottery with worst outcome x_m. Expressing the definition of $CCE_j(P)$, or CCE_j for short, ($j = A, B$), in terms of CT yields: $v_j(x_m) + (1 - \lambda)u_j(CCE_j - x_m) = v_j(x_m) + (1 - \lambda)\sum_{i=1}^{l} p_i u_j(x_i - x_m)$ which implies $u_j(CCE_j - x_m) = \sum_{i=1}^{l} p_i u_j(x_i - x_m)$. As the CCE_j is purely defined by the EU-component of CT based o u, we know from Pratt (1964) that we always have $CCE_A - x_m < CCE_B - x_m$, i.e., $CCE_A < CCE_B$ if and only if $u''_A(x)/u'_A(x) > u''_B(x)/u'_B(x)$ for all $x > 0 \in \mathbb{R}$.

Expressing the definition of $CE_j(P)$, or CE_j for short, ($j = A, B$), in terms of CT yields: $v_j(CE_j) = v_j(x_m) + \sum_i p_i u_j(x_i - x_m)$. This implies

$$\frac{v_A(CE_A) - v_A(x_m)}{\sum_{i=1}^{l} p_i u_A(x_i - x_m)} = 1 = \frac{v_B(CE_B) - v_B(x_m)}{\sum_{i=1}^{l} p_i u_B(x_i - x_m)}, \tag{15}$$

which can be expressed as

$$\frac{\int_{x_m}^{CE_A} v'_A(y)dy}{u_A(CCE_A - x_m)} = \frac{\int_{x_m}^{CE_B} v'_B(y)dy}{u_B(CCE_B - x_m)}. \tag{16}$$

First we assume that $u''_A(x)/u'_A(x) > u''_B(x)/u'_B(x)$ and $v'_A(y)/u_A(x) > v'_B(y)/u_B(x)$ for all $y, x > 0 \in \mathbb{R}$ holds. We prove that $CE_A < CE_B$. From $u''_A(x)/u'_A(x) > u''_B(x)/u'_B(x)$ for all $x > 0 \in \mathbb{R}$ it follows (see above) that $CCE_A < CCE_B$. Consider C_A defined by

$$\frac{\int_{x_m}^{C_A} v_A'(y)dy}{u_A(CCE_B - x_m)} = 1 = \frac{\int_{x_m}^{CE_B} v_B'(y)dy}{u_B(CCE_B - x_m)}. \tag{17}$$

Since $v_A'(y)/u_A(CCE_B - x_m) > v_B'(y)/u_B(CCE_B - x_m)$ for all $y \in \mathbb{R}$, we must have $C_A < CE_B$. As

$$\frac{\int_{x_m}^{C_A} v_A'(y)dy}{u_A(CCE_B - x_m)} = 1 = \frac{\int_{x_m}^{CE_A} v_A'(y)dy}{u_A(CCE_A - x_m)} \tag{18}$$

and $u_A(CCE_A - x_m) < u_A(CCE_B - x_m)$ we must have $CE_A < C_A < CE_B$. Thus, $CE_A < CE_B$ follows as desired.

Next, we assume that $CE_A(P) < CE_B(P)$ for all nondegenerate lotteries P and prove that $v_A'(y)/u_A(x) > v_B'(y)/u_B(x)$ for all $x > 0, y \in \mathbb{R}$ (as $u_A''(x)/u_A'(x) > u_B''(x)/u_B'(x)$ has already been shown). For an arbitrary probability $0 < p < 1$ and arbitrary outcomes $x > 0, y \in \mathbb{R}$, consider the lottery $P = (p, y + x, 1 - p, y)$. Obviously we have $v_A(CE_A(P)) = v_A(y) + pu_A(x)$ which yields

$$\frac{v_A(CE_A(P)) - v_A(y)}{pu_A(x)} = 1. \tag{19}$$

Analogously we get

$$\frac{v_B(CE_B(P)) - v_B(y)}{pu_B(x)} = 1. \tag{20}$$

This implies

$$\frac{\int_{y}^{CE_A} v_A'(z)dz}{pu_A(x)} = 1 = \frac{\int_{y}^{CE_B} v_B'(z)dz}{pu_B(x)}. \tag{21}$$

Suppose now that p converges to zero. This implies that CE_A and CE_B converge to y. This follows from the fact that CT is continuous in probabilities. Now $CE_A < CE_B$ can only hold if $v_A'(y)/u_A(x) > v_B'(y)/u_B(x)$. As this argument is valid arbitrary $x > 0, y \in \mathbb{R}$, it is also valid for all

$x > 0, y \in \mathbb{R}$.

This completes the proof of Proposition 4. $\square$

Proof of Proposition 5: The proof follows immediately from the arguments provided preceding the proposition in the main text. $\square$

Proof of Lemma 6: First we note that by Debreu (1954) the preference conditions imply the existence of a strongly monotonic continuous function $W : \mathbb{R}^n \to \mathbb{R}^n$ that represents the preference $\succcurlyeq$ on $\mathcal{F}$. Obviously, W also represents $\succcurlyeq$ on each set $\mathcal{F}_m, m = 1, \ldots, n$.

Let us fix an arbitrary state $m \in \mathcal{S}$. Without loss of generality let $m = 1$. Continuity and monotonicity imply that for each act $f \in \mathcal{F}_1$ there exists a unique outcome x^f such that $f \sim (f_1, x^f, \ldots, x^f)$. Next we restrict the analysis to the set of quasi binary acts $\{f \in \mathcal{F}_1 | (f_1, x^f, \ldots, x^f)\}$. In our notation for acts (i.e., $f = (1 : f_1; f - f_1)$) this set is isomorphic to the two dimensional set $F_1 := \{f \in \mathcal{F}_1 | (1 : f_1; x^f - f_1)\}$ ($\cong \mathbb{R} \times \mathbb{R}_+$). The restriction of the preference $\succcurlyeq$ to F_1, which for simplicity we also denote $\succcurlyeq$, inherits weak order, continuity and monotonicity from $\succcurlyeq$ on $\mathcal{F}_1$. Additionally, it satisfies the triple cancellation condition formulated with indifferences on F_1. In the presence of weak order, monotonicity and continuity, and the structural richness that we have in F_1, the indifference version of triple cancellation is equivalent to the preference version of the triple cancellation (see Köbberling and Wakker, 2003, for a similar argument showing that their indifference version of tradeoff consistency is equivalent to the preference version of tradeoff consistency). Hence, by Corollary 3.6 and Remark 3.7 of Wakker (1993a) it follows that there exists (jointly cardinal) continuous and strictly increasing functions $V_1 : \mathbb{R} \to \mathbb{R}$ and $U_{-1} : \mathbb{R}_+ \to \mathbb{R}$ such that $\succcurlyeq$ on F_1 is represented by

$$W_1(f) = V_1(f_1) + U_{-1}(x^f - f_1).$$

Note that continuity and monotonicity imply that $U_{-1}(0) = 0$. This means that, for $a > 0$ and b real, V_1 can be replaced by $aV_1 + b$ whenever U_{-1} is replaced by aU_{-1}. Next we extend U_{-1}

from $\mathbb{R}_+$ to $\{0\} \times \mathbb{R}_+^{n-1}$. Using the indifference $f \sim (1 : f_1; 0, x^f - f_1, \ldots, x^f - f_1)$ we can define $U_{-1} : \{0\} \times \mathbb{R}_+^{n-1} \to \mathbb{R}$ through

$$U_{-1}(0, f - f_1) := U_{-1}(0, x^f - f_1, \ldots, x^f - f_1) \equiv U_{-1}(0, x^f - f_1).$$

This way continuity and strong monotonicity of U_{-1} is inherited through x^f. Then, for $f, g \in \mathcal{F}_1$, we have

$$\begin{aligned} f \succcurlyeq g &\Leftrightarrow (f_1, x^f - f_1) \succcurlyeq (g_1, x^g - g_1) \\ &\Leftrightarrow V_1(f_1) + U_{-1}(x^f - f_1) \geq V_1(g_1) + U_{-1}(x^g - g_1) \\ &\Leftrightarrow V_1(f_1) + U_{-1}(f - f_1) \geq V_1(g_1) + U_{-1}(g - g_1), \end{aligned}$$

demonstrating that $W_1(f) = V_1(f_1) + U_{-1}(f - f_1)$ represents $\succcurlyeq$ on $\mathcal{F}_1$. Obviously, the uniqueness results for V_1 and U_{-1} are maintained through this extension of the representation.

In the preceding analysis we have fixed state $m = 1$. The proof for any arbitrary state $m \in \mathcal{S}$ is completely analogous. Hence, we can conclude that for each state $m \in \mathcal{S}$ there exists strongly monotonic and continuous functions $V_m : \mathbb{R} \to \mathbb{R}, U_{-m} : \{0\} \times \mathbb{R}_+^{n-1} \to \mathbb{R}$ such that on each set $\mathcal{F}_m$ the preference $\succcurlyeq$ is represented by

$$W_m(f) = V_m(f_m) + U_{-m}(f - f_m).$$

The functions V_m, U_{-m} satisfy the corresponding uniqueness results for the representation W_m on $\mathcal{F}_m$, for all $m \in \mathcal{S}$.

Next we apply once more WOS. We take two arbitrary but distinct states $m, m' \in \mathcal{S}$. Locally, in a small neighborhood, we can find for all α, β, γ, δ and f, g, such that the following three indifferences hold: $(m : \alpha; f) \sim (m : \beta; g)$, $(m : \gamma; f) \sim (m : \delta; g)$ and $(m' : \alpha; f') \sim (m' : \beta; g')$ and all acts are from

$\mathcal{F}_m \cap \mathcal{F}_{m'}$. By WOS it follows that $(m' : \gamma; f') \sim (m' : \delta; g')$ with these acts being from $\mathcal{F}_m \cap \mathcal{F}_{m'}$. As both W_m and $W_{m'}$ represent preferences on $\mathcal{F}_m \cap \mathcal{F}_{m'}$ these functions must be ordinal transformations of each other. Further, substitution of W_m in the former two indifferences and taking differences of the resulting equations and cancelling common terms, implies $V_m(\alpha) - V_m(\beta) = V_m(\gamma) - V_m(\delta)$. Similarly, substitution of $W_{m'}$ in the latter two indifferences, implies $V_{m'}(\alpha) - V_{m'}(\beta) = V_{m'}(\gamma) - V_{m'}(\delta)$. As, α, β and γ, δ were arbitrary, it follows that V_m and $V_{m'}$ are (first locally and by continuity also globally) proportional. As $W_m = V_m$ and $W_{m'} = V_{m'}$ for all constant acts, and the latter are included in $\mathcal{F}_m \cap \mathcal{F}_{m'}$, it follows that W_m and $W_{m'}$ are, actually, cardinally related. Hence, we can choose them identical on the set of common acts $\mathcal{F}_m \cap \mathcal{F}_{m'}$. In particular this means that $U_{-m} = U_{-m'}$ on $(\mathbb{R}_+^{n-m} \times \{0\} \times \rightarrow \mathbb{R}_+^{m-1}) \cap (\mathbb{R}_+^{n-m'} \times \{0\} \times \rightarrow \mathbb{R}_+^{m'-1})$. As m and m' were arbitrary chosen, we can set $v := V_s$ for all $s \in \mathcal{S}$. That is

$$W_s(f) = v(f_s) + U_{-s}(f - f_s)$$

holds on each set $\mathcal{F}_s$. Further, uniqueness results are maintained for v and $U_{-s}, s \in \mathcal{S}$. This completes the proof of Lemma 6. □

Proof of Theorem 7: First we prove that statement (ii) implies statement (i). Monotonicity of $\succcurlyeq$ follows from the strong monotonicity of W. Continuity of $\succcurlyeq$ follows from the continuity of W. The fact that W is representing $\succcurlyeq$ on $\mathcal{F}$ implies that $\succcurlyeq$ is a weak order. Notice that on each set $\mathcal{F}_s$ the act f is evaluated by $W(f) = v(f_s) + U(f - f_s)$, where $U(f - f_s) = \sum_{t \in \mathcal{S} \setminus \{s\}} \pi_t u(f_t - f_s)$, which is an additively separable representation of $\succcurlyeq$ on $\mathcal{F}_s$ and has v independent of s. Hence, substitution of W for the indifferences in the definition of WOS immediately shows that $\succcurlyeq$ satisfies WOS. To derive tradeoff consistency for chance assume that the acts in the following indifferences are all from the same set $\mathcal{F}_m$ for some state $m \in \mathcal{S}$, and that

$$\begin{aligned} (f_m + w)_t f &\sim (f_m + x)_t g, \\ (f_m + y)_t f &\sim (f_m + z)_t g \\ \text{and } (f'_m + w)_{t'} f' &\sim (f'_m + x)_{t'} g' \text{ hold} \\ \text{but } (f'_m + y)_{t'} f' &\succ (f'_m + z)_{t'} g' \end{aligned}$$

for some $t, t' \neq m$. Substitution of W into the first two indifferences implies that

$$\begin{aligned} v(f_m) + \sum_{r \in \mathcal{S} \setminus \{m,t\}} \pi_r u(f_r - f_m) + \pi_t u(w) &= v(f_m) + \sum_{r \in \mathcal{S} \setminus \{m,t\}} \pi_r u(g_r - f_m) + \pi_t u(x) \\ &\quad \text{and} \\ v(f_m) + \sum_{r \in \mathcal{S} \setminus \{m,t\}} \pi_r u(f_r - f_m) + \pi_t u(y) &= v(f_m) + \sum_{r \in \mathcal{S} \setminus \{m,t\}} \pi_r u(g_r - f_m) + \pi_t u(z) \end{aligned}$$

hold. Subtracting the second equation from the first and cancelling common terms gives:

$$\pi_t[u(w) - u(x)] = \pi_t[u(y) - u(z)]$$

or

$$u(w) - u(x) = u(y) - u(z)$$

as the probability π_t is positive.

Substitution of W into the third indifference and the latter preference gives, by using similar calculus,

$$u(w) - u(x) < u(y) - u(z),$$

a contradiction.

If in the previous analysis, instead of $(f'_m + y)_{t'} f' \succ (f'_m + z)_{t'} g'$, we assume $(f'_m + y)_{t'} f' \prec$

$(f'_m + z)_{t'}g'$, a similar contradiction (i.e., $u(w) - u(x) = u(y) - u(z)$ and $u(w) - u(x) > u(y) - u(z)$) is obtained. Hence, $(f'_m + y)_{t'}f' \sim (f'_m + z)_{t'}g'$ must hold. As $m \in \mathcal{S}$ was arbitrary, it follows that tradeoff consistency for chance holds on each set $\mathcal{F}_s$. Hence, it holds on $\mathcal{F}$.

Finally, the property that for all $y \in \mathbb{R}, x > \varepsilon > 0$

$$v(y + \varepsilon) - v(y) > u(x) - u(x - \varepsilon),$$

holds follows from the strong monotonicity property of W and Lemma 6.

Next we assume statement (ii) and derive statement (i). The assumptions of Lemma 6 are satisfied, hence, there exists continuous strongly monotonic functions $v : \mathbb{R} \to \mathbb{R}, U_{-s} : \mathbb{R}_+^{n-s} \times \{0\} \times \to \mathbb{R}_+^{s-1}$ with $U_{-s}(0, \ldots, 0) = 0$, $s \in \mathcal{S}$, such that the preference $\succcurlyeq$ is represented by

$$W(f) = v(f_s) + U_{-s}(f - f_s)$$

for $f \in \mathcal{F}_s$. Further, for $a > 0$ and real-valued b, the function v can be replaced by $av + b$ whenever U_{-s} is replaced by aU_{-s}, $s \in \{1, \ldots, n\}$.

Take any arbitrary state $m \in \mathcal{S}$. Tradeoff consistency for chance implies that if

$$\begin{aligned} (f_m + w)_t f &\sim (f_m + x)_t g, \\ (f_m + y)_t f &\sim (f_m + z)_t g \\ \text{and } (f'_m + w)_{t'} f' &\sim (f'_m + x)_{t'} g' \text{ hold,} \end{aligned}$$

then $(f'_m + y)_{t'}f' \sim (f'_m + z)_{t'}g'$ follows, provided that $t, t' \neq m$ and all acts involved are from the set

$\mathcal{F}_m$. Substituting $W = v + U_{-m}$ we obtain that on $\mathbb{R}_+^{n-m} \times \{0\} \times \to \mathbb{R}_+^{m-1}$ the equalities

$$\begin{aligned} U_{-m}((f_m + w)_t f - f_m) &= U_{-m}((f_m + x)_t g - f_m) \\ U_{-m}((f_m + y)_t f - f_m) &= U_{-m}((f_m + z)_t g - f_m) \\ \text{and } U_{-m}((f'_m + w)_{t'} f' - f'_m) &= U_{-m}((f'_m + x)_{t'} g' - f'_m) \\ \text{imply } U_{-m}((f'_m + y)_{t'} f' - f'_m) &= U_{-m}((f'_m + z)_{t'} g' - f'_m). \end{aligned}$$

This condition is analogous to tradeoff consistency (see Köbberling and Wakker 2003) for the function U_{-m} (representing a continuous monotonic preference) on $\mathbb{R}_+^{n-m} \times \{0\} \times \to \mathbb{R}_+^{m-1}$. Following Köbberling and Wakker (2003) this implies that there exist positive numbers $\rho_{-m,t}, t \in \mathcal{S} \backslash \{m\}$ and a continuous strictly increasing utility function $u_{-m} : \mathbb{R}_+ \to \mathbb{R}$ such that

$$U_{-m}(f - f_m) = \sum_{t \in \mathcal{S} \backslash \{m\}} \rho_{-m,t} u_{-m}(f_t - f_m).$$

From $U_{-m}(0) = 0$ it follows that $u_{-m}(0) = 0$. This means that $\succcurlyeq$ on $\mathcal{F}_m$ is represented by

$$W(f) = v(f_m) + \sum_{t \in \mathcal{S} \backslash \{m\}} \rho_{-m,t} u_{-m}(f_t - f_m).$$

We now set $v(0) = 0$ to fix the location of v.

Recall that $m \in \mathcal{S}$ was arbitrary chosen. Hence, we conclude that for each state $s \in \mathcal{S}$ there exist strictly increasing and continuous function $u_{-s} : \mathbb{R}_+ \to \mathbb{R}$ with $u_{-s}(0) = 0$, and positive numbers $\rho_{-s,t}, t \in \mathcal{S} \backslash \{s\}$ such that $W_s(f) := v(f_s) + \sum_{t \in \mathcal{S} \backslash \{s\}} \rho_{-s,t} u_{-s}(f_t - f_s)$ represents the preference on the set of acts $\mathcal{F}_s$. Further the functions v and u_{-s} are unique up to multiplication by a positive constant (i.e., they are ratio scales).

Take any two distinct states $s, s' \in \mathcal{S}$ and consider the restriction of $\succcurlyeq$ on the set of acts $\mathcal{F}_{\{s,s'\}} := \mathcal{F}_s \cap \mathcal{F}_{s'}$. On this set both W_s and $W_{s'}$ represent $\succcurlyeq$. Uniqueness results imply that $u_{-s'} = u_{-s}$ (as both have v in common) and that $\rho_{-s,t} = \rho_{-s',t} =: \rho_t$ for all $t \in \mathcal{S}\backslash\{s, s'\}$. As $s' \in \mathcal{S}$ was chosen arbitrary it follows that the positive numbers $\rho_{-s,t}$ are independent of $s \in \mathcal{S}$, such that we have n positive numbers $\rho_t, t \in \mathcal{S}$.

We define $\tilde{u} := u_{-s}$. Further, we set $\rho = \sum_{t\in\mathcal{S}} \rho_t$ and define

$$\pi_t := \frac{\rho_t}{\rho} \text{ for each } t \in \mathcal{S}$$

and obtain a subjective probability distribution over the states in $\mathcal{S}$. Finally, we define $u := \rho\tilde{u}$.

Hence, we have shown that the representations W of $\succcurlyeq$ on $\mathcal{F}$ is of the form

$$W(f) = v(f_s) + \sum_{t\in\mathcal{S}\backslash\{s\}} \pi_t u(f_t - f_s) \text{ for } f \in \mathcal{F}_s, s \in \mathcal{S}.$$

Hence, we have derived statement (i) of the theorem.

By construction the probabilities $\pi_s, s \in \mathcal{S}$ are uniquely determined. By construction, for positive a and some constant b, we can replace v and u by $av + b$ and au, respectively. That there is no further flexibility in the choice of these functions follows from Lemma 6. This concludes the proof of Theorem 7. □

References

Allais, Maurice (1953), "Le Comportement de l'Homme Rationnel devant le Risque: Critique des Postulats et Axiomes de l'Ecole Américaine," *Econometrica* 21, 503–546.

Alon, Shiri (2013), "Derivation of a Cardinal Utility Through a Weak Trade-off Consistency Requirement," *Mathematics of Operations Research*, forthcoming (moor.2013.0615).

Andreoni, J, and C. Sprenger (2010), "Certain and Uncertain Utility: The Allais Paradox and Five Decision Theory Phenomena," Working Paper.

Andreoni, J, and C. Sprenger (2012), "Risk Preferences Are Not Time Preferences," *American Economic Review* 102, 3357-3376.

Andreoni, J. and W. Harbaugh (2010), "Unexpected Utility: Five Experimental Tests of Preferences For Risk," Working Paper.

Arrow, K.J. (1965), "The theory of risk aversion". In: *Aspects of the Theory of Risk Bearing.* Yrjo Jahnssonin Saatio, Helsinki. Reprinted in: *Essays in the Theory of Risk Bearing* (1971, pp. 90–109). Markham Publ. Co., Chicago.

Arrow, Kenneth J., and Hurwicz, Leonid (1972) "Decision Making Under Ignorance," in C. F. Carter and J.L. Ford (eds.), *Uncertainty and Expectations in Economics. Essays in Honour of G.L.S. Shackle.* Oxford: Basil Blackwell.

Bernoulli, Daniel (1954), "Exposition of a New Theory on the Measurement of Risk," *Econometrica* 22, 23–36. Translated version of Bernoulli, Daniel (1738), "Specimen Theoriae Novae de Mensura Sortis," Commentarii Academiae Scientiarum Imperialis Petropolitanae, Tomus V [Papers of the Imperial Academy of Sciences in Petersburg, Vol. V], 175–192.

Bleichrodt, Han and Ulrich Schmidt (2002), "A Context-Dependent Model of the Gambling Effect," *Management Science* 48, 802–812.

Camerer, Colin F. (1992), "Recent Tests of Generalizations of Expected Utility Theory." In Ward Edwards (ed.) *Utility Theories: Measurement and Applications*, 207–251, Kluwer Academic Publishers, Dordrecht.

Cerreia-Vioglio, Simone, David Dillenberger, and Pietro Ortoleva (2013), "Cautious Expected Utility and the Certainty Effect,"Working Paper.

Chateauneuf, Alain, Jürgen Eichberger and Simon Grant (2007), "Choice under Uncertainty with the Best and Worst in Mind: NEO-Additive Capacities," *Journal of Economic Theory* 137, 538–567.

Cohen, Michèle (1992), "Security Level, Potential Level, Expected Utility: A Three-Criteria Decision Model under Risk," Theory and Decision 33, 101–134.

Cohen, M. and J.Y. Jaffray (1988) "Preponderance of the Certainty Effect over Probabillity Distortion in Decision Making under Risk," in B.R. Munier (ed), *Risk, Decision, and Rationality.* D. Reidel: Dordrecht.

Conlisk, John (1989), "Three Variants on the Allais Example," *American Economic Review* 79, 392–407.

Conlisk, John (1993), "The Utility of Gambling," *Journal of Risk and Uncertainty* 6, 255–275.

Debreu, Gerard (1954), "Representation of a Preference Ordering by a Numerical Function," in R.M. Thrall, C.H. Coombs, R.L. Davis (Eds.), *Decision Processes* 159–165, Wiley, New York.

Debreu, Gerard (1960), "Topological Methods in Cardinal Utility Theory," in K.J. Arrow, S. Karlin, P. Suppes (Eds.), *Mathematical Methods in the Social Sciences*, 16–26, Stanford University Press, Stanford, CA.

Diecidue, Enrico, Ulrich Schmidt, and Peter P. Wakker (2004),"The Utility of Gambling Reconsidered," *Journal of Risk and Uncertainty* 29, 241–259.

Dillenberger, David (2010), "Preferences for One-Shot Resolution of Uncertainty and Allais-Type Behavior," *Econometrica* 78, 1973–2004.

Dyer, James S., and Rakesh K. Sarin (1982), "Relative Risk Aversion," *Management Science* 28, 875–886.

Ellsberg, Daniel (1961), "Risk, Ambiguity and the Savage Axioms," *Quarterly Journal of Economics* 75, 643–669.

Fishburn, Peter C. (1977), "Mean-Risk Analysis with Risk Associated with Below-Target Returns," *American Economic Review* 67, 116–126.

Fishburn, Peter C. (1980), "A Simple Model for the Utility of Gambling," *Psychometrika* 45, 435–448.

Friedman, Milton, and Leonard J. Savage (1948), "The Utility Analysis of Choices Involving Risk," *Journal of Political Economy* 56, 279–304.

Gilboa, Itzhak (1988), "A Combination of Expected Utility and Maxmin Decision Criteria," *Journal of Mathematical Psychology* 32, 405–420.

Gilboa, Itzhak and David Schmeidler (1989), "Maxmin Expected Utility with a Non-Unique Prior," *Journal of Mathematical Economics* 18, 141–153.

Gneezy, Uri, John List, and George Wu (2006), "The Uncertainty Effect: When a Risky Prospect Is Valued Less than Its Worst Possible Outcome," *Quarterly Journal of Economics* 121, 1283–1309.

Gul, Faruk (1991), "A Theory of Disappointment Aversion," *Econometrica* 59, 667–686.

Harless, and Camerer (1994), "The Predictive Utility of Generalized Expected Utility Theories," *Econometrica* 62, 1251–1289.

Hey, John and Chris Orme (1994), "Investigating Generalizations of Expected Utility Theory Using Experimental Data," *Econometrica* 62, 1291–1326.

Jaffray, Jean Yves (1988), "Choice Under Risk and the Security Factor: An Axiomatic Model," *Theory and Decision*, 24, 169–200.

Kahneman, Daniel and Amos Tversky (1979), "Prospect Theory: An Analysis of Decision under Risk," *Econometrica* 47, 263–291.

Kimball, Miles S. (1990): "Precautionary Saving in the Small and in the Large", *Econometrica* 58, 53–73.

Köbberling, Veronika and Peter P. Wakker, (2003), "Preference Foundations for Nonexpected Utility: A Generalized and Simplified Technique." *Mathematics of Operations Research* 28, 395–423.

Krantz, David H., R. Duncan Luce, Patrick Suppes, and Amos Tversky, (1971), *Foundations of Measurement, Vol. I (Additive and Polynomial Representations).* Academic Press, New York. 2nd edn. 2007, Dover Publications, New York.

Lopes, Lola L. (1987), "Between Hope and Fear: The Psychology of Risk. In Leonard Berkowitz (Ed.), *Advances in Experimental and Social Psychology* (pp. 255–295), San Diego, Academic Press.

Luce, R. Duncan, and Anthony A.J. Marley (2000) "On Elements of Chance," *Theory and Decision* 49, 97–126.

Luce, R. Duncan and Howard Raiffa (1957), *Games and Decisions*, Wiley, New York.

Luce, R. Duncan and John W. Tukey (1964), "Simultaneous Conjoint Meassurement: A New Type of Fundamental Measurement," *Journal of Mathematical Psychology* 1, 1–27.

Payne, John W., Dan J. Laughhunn, and Roy L. Crum (1980), "Translation of Gambles and Aspiration Level Effects in Risky Choice Behavior," *Management Science* 26, 1039–1060.

Payne, John W., Dan J. Laughhunn, and Roy L. Crum (1981), "Further Tests of Aspiration Level Effects in Risky Behavior," *Management Science* 27, 953–958.

Pratt, J.W. (1964). "Risk Aversion in the Small and in the Large," *Econometrica 32*, 122–136.

Quiggin, John (1981), "Risk Perception and Risk Aversion among Australian Farmers," *Australian Journal of Agricultural Economics* 25, 160–169.

Quiggin, John (1982), "A Theory of Anticipated Utility," *Journal of Economic Behaviour and Organization* 3, 323–343.

Rabin, Matthew (2000), "Risk Aversion and Expected-utility Theory: A Calibration Theorem," *Econometrica* 68, 1281–1292.

Rawls, J (1971), *A Theory of Justice.* Cambridge, Mass.

Rothschild, Michael and Joseph E. Stiglitz (1970), "Increasing Risk: I. A Definition," *Journal of Economic Theory* 2, 225–243.

Roy, Andrew D. (1952) "Safety First and the Holding of Assets," *Econometrica* 20, 431–449.

Savage, Leonard J., (1954), *The Foundations of Statistics.* New York: Wiley.

Schmeidler, David (1989), "Subjective Probability and Expected Utility without Additivity," *Econometrica* 57, 571–587.

Schmidt, Ulrich (1998), "A Measurement of the Certainty Effect," *Journal of Mathematical Psychology* 42, 32–47.

Segal, Uzi and Avia Spivak (1990), "First Order Versus Second Order of Risk Aversion," *Journal of Economic Theory* 51, 111–125.

Simonsohn, Uri (2009), "Direct-Risk-Aversion: Evidence from Risky Prospects Valued Below Their Worst Outcome," *Psychological Science* 20, 686–692.

Spinu, Vitalie (2012), "From Simple to Complex: A General Method for Extending Behavioral Foundations to Infinitely Many Outcomes," Working Paper, Erasmus University Rotterdam, the Netherlands.

Starmer, Chris (2000), "Developments in Non- Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk," *Journal of Economic Literature* 38, 332–382.

Starmer, Chris and Robert Sugden (1989), "Violations of the Independence Axiom in Common Ratio Problems: An Experimental Test of some Competing Hypotheses," *Annals of Operations Research* 19, 79–102.

Tversky, Amos and Daniel Kahneman (1992), "Advances in Prospect Theory: Cumulative Representation of Uncertainty," *Journal of Risk and Uncertainty* 5, 297–323.

Wakker, Peter P., (1989), *Additive Representations of Preferences, A New Foundation of Decision Analysis*, Kluwer Academic Publishers, Dordrecht.

Wakker, Peter P., (1993a), "Additive Representations on Rank-ordered Sets II. The Topological Approach," *Journal of Mathematical Economics* 22, 1–26.

Wakker, Peter P., (1993b), "Unbounded Utility for Savage's "*Foundation of Statistics*," and Other Models." *Mathematics of Operations Research* 18, 446–485.

Wakker, Peter P. (2010), *Prospect Theory for Risk and Ambiguity*, Cambridge University Press, Cambridge, UK.

Webb, Craig S., and Horst Zank (2011). "Accounting for Optimism and Pessimism in Expected Utility," *Journal of Mathematical Economics* 47, 706–717.

Yaari, Menahem E. (1987), "The Dual Theory of Choice under Risk," *Econometrica* 55, 95–115.