A Simple Model of Cumulative Prospect Theory

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Abstract. The present paper combines loss attitudes and linear utility by providing an axiomatic analysis of cumulative prospect theory in the framework for decision under uncertainty. We derive a two-sided variant of Choquet expected utility with possibly different capacities for gains and for losses, and linear utility. Naturally, utility may have a kink at the status quo, which allows for the exhibition of loss aversion. The central condition of our model is termed independence of common increments.

Keywords: comonotonic sure thing principle, cumulative prospect theory, linear utility, loss aversion.

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1 Introduction

Empirical research has shown that expected utility theory (EU) fails to provide a good description of individual behavior in situations of uncertainty. Examples are the famous paradoxes of Allais (1953) and Ellsberg (1961). This evidence has motivated the development of alternative theories (the so-called non-expected utility theories), which allow for the exhibition of "paradoxical behavior." Building upon its predecessor prospect theory (Kahneman and Tversky, 1979), cumulative prospect theory (CPT) has nowadays become one of the most prominent of these alternatives (Starmer 2000).

Recently, a new criticism of EU has been put forward by Rabin (2000) and Rabin and Thaler (2001). Following earlier work by Hansson (1988), these authors show that reasonable degrees of risk aversion over small and modest stakes imply unreasonable high degrees of risk aversion over large stakes in the EU framework. For instance an EU-maximizer, who initially rejects a 50-50 bet of loosing \$10 and winning \$11 at any wealth level, would also reject any 50-50 bet of losing \$100 and winning \$$x$ for an arbitrarily large value of x. Since this high degree of risk aversion seems to be irrational, Rabin (2000) concluded that EU is only a good representation of risk neutral behavior, which means that utility has to be linear. Neilson (2001) has shown that this criticism on EU carries over to rank-dependent utility (RDU), a further alternative to EU and a precursor of CPT. More precisely, in the rank-dependent utility framework the utility function should also be linear because a concave utility implies, as for EU, unreasonable high degrees of risk aversion over large stakes.

What drives these two results is the possibility to represent choice behavior by a mathematical functional where attitudes towards uncertainty can entirely be separated from attitudes towards outcomes. Indeed, under EU, the decision weight by which the utility of an outcome is multiplied, equals the probability of occurrence of the corresponding outcome or associated event. Obviously, this probability is unrelated to the magnitude of the outcome. Under RDU the decision weight reveals information about the probability and further about the rank of an outcome compared to other possible payoffs. Again no dependence of the decision weight and the magnitude of outcomes exists. For CPT, in addition to the rank of an outcome, the sign of the outcome relative to the status quo is influencing the decision weights, but beyond that no dependence on the magnitude of outcomes holds.

It appears therefore that in order to avoid the Rabin paradoxical behavior, but at the same time maintain the independence of decision weights and the magnitude of outcomes, utility in the above described models has to be linear. Considering this strong implication, the goal of the present paper is to investigate linear utility for decision under uncertainty by providing an axiomatic analysis of CPT.

Because CPT combines three desirable features (rank-dependence, reference-dependence, and sign-dependence) with theoretical tractability, it is currently seen as the most promising decision model (see Starmer 2000, page 370). The model was first proposed by Starmer and Sugden (1989). Later, axiomatizations of CPT have been provided by Luce and Fishburn (1991), Tversky and Kahneman (1992), Wakker and Tversky (1993), Chateauneuf and Wakker (1999), and Schmidt (2003). This paper provides a new axiomatization of CPT with a piecewise linear utility function. More precisely, utility is linear for gains and linear for losses with a possible kink at the status quo. If loss aversion is satisfied, utility is steeper on the domain of losses than it is on the domain of gains.

Linear utility has a long tradition in theoretical and empirical research. An axiomatic foundation of subjective expected utility with linear utility was provided by de Finetti (1931). Preston and Baratta (1948) used a linear utility model in order to estimate probability distortions. Edwards (1955) reports about a series of experiments which support our model. He finds evidence for sign-dependent probability distortions and also for linear utility. Many other studies observed linear utility for losses (Hershey and Schoemaker 1980, Schneider and Lopes 1986, Cohen, Jaffray, and Said 1987, Weber and Bottom 1989, Lopes and Oden 1999). Generally, for small stakes the evidence suggests that utility is linear (Lopes 1995, Fox, Rogers, and Tversky 1996, Kilka and Weber 2001).

Handa (1977) axiomatized a model of subjective expected value, which was implicitly used by Preston and Baratta (1948) and already discussed in Edwards (1955). A model for decision under risk that combines linear utility and distorted probabilities is the dual theory (DT) of Yaari (1987). Such a model has been analyzed in Safra and Segal (1998) for a restricted class of weighting functions. For decision under uncertainty the model of Chateauneuf (1991) provides a Choquet expected utility form with linear utility. A similar model is presented in de Waegenaere and Wakker (2001). A further study by Diecidue and Wakker (2002) provides an axiomatization of linear utility and decision weights in the framework of de Finetti. In all of these studies, reference-dependence and therefore also loss aversion is excluded.

Also, linear utility has often been employed in economic applications. Some examples are firm behavior under risk (Demers and Demers 1990), insurance demand (Doherty and Eeckhoudt 1995, Schmidt 1996), insurance pricing (Wang 1995, 1996, Wang, Young and Panjer 1997), agency theory (Schmidt 1999a), the equity premium puzzle (Epstein and Zin 1990) and efficient risk-sharing (Schmidt 1999b). We are convinced that CPT with linear utility may generate new insights in such theoretical applications. Interestingly some applications of our model to investment behavior have already been conducted (Barberis, Huang, and Santos 2001, Barberis and Huang 2001, Roger 2001). Also, van der Hoek and Sherris (2001) propose a risk measure for portfolio choice and insurance decisions based on DT and choose different weighting functions for gains and losses. Therefore, our model can serve as a theoretical basis for their risk measure through the additional freedom gained by reference- and sign-dependence.

Our paper is organized as follows. Some preliminary notation is introduced in the next section. Then, in Section 3, a first representation theorem is presented for the case of a finite state space. These results are then extended to more general state spaces in Sections 4 and 5. Concluding remarks are presented in Section 6. The appendix contains proofs of the main theorems.

2 Notation

Let S be a (finite or infinite) set of states of the world. Subsets of S will be denoted by $A, B, \ldots$; the complement of A (with respect to S) is denoted by A^c. The state space is endowed with an *algebra* $\mathcal{A}$ of subsets of S. Therefore, (i) $S \in \mathcal{A}$, (ii) if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$, and (iii) if $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$. Subsets of S which are contained in $\mathcal{A}$ are called *events*. A (finite) *partition* $\{A_1, \ldots, A_n\}$ of S is a collection of disjoint events, the union of which equals S.

The set of outcomes is $I\!R$, indicating money. Members of the outcome set are denoted by $x, y, z, \ldots$. An act $f : S \to I\!R$, $s \mapsto f(s)$ assigns to each state an outcome. We assume throughout that acts are *bounded* (i.e., for any act f there exists $z \in I\!R$ such that $|f(s)| \leqslant z$ for all states s) and *measurable* (i.e., the inverse of each interval is an event).

The set of all acts is denoted by $\mathcal{F}$. An important subset of $\mathcal{F}$ is the set of simple acts, $\mathcal{F}^s$. Simple acts take only finitely many values. Therefore, for $f \in \mathcal{F}^s$ there exists a partition $\{A_1, \ldots, A_n\}$ such that $f = \sum_{i=1}^{n} x_i 1_{A_i}$, where 1_{A_i} is the indicator function of event A_i. It is understood that the act f assigns outcome x_i for states $s \in A_i$, $i = 1, \ldots, n$.

We use the notation $f_A g$ for an act that agrees with act f on event A and with act g on the complement A^c. Also we use $h_i f$ instead of $h_{\{s_i\}} f$ for some state $s_i \in S$. Sometimes we identify constant acts with the corresponding outcome. We may thus write $f_A x$ for an act agreeing with f on A and giving outcome x for states $s \in A^c$; similarly we use $x_A f$.

We assume a preference relation $\succcurlyeq$ on the set of acts. As usually, the statement $f \succcurlyeq g$ means that act f is weakly preferred to act g. The symbols $\succ$ and $\sim$ denote strict preference and indifference, respectively. Sometimes we write $f \preccurlyeq g$ ($f \prec g$) instead of $g \succcurlyeq f$ ($g \succ f$). The preference relation $\succcurlyeq$ is a *weak order* if it is *complete* ($f \succcurlyeq g$ or $g \succcurlyeq f$ for any acts f, g) and *transitive* ($f \succcurlyeq g$ and $g \succcurlyeq h$ implies $f \succcurlyeq h$). A functional $V : \mathcal{F} \to I\!R$ represents the preference relation $\succcurlyeq$ if for all $f, g \in \mathcal{F}$ we have $f \succcurlyeq g \Leftrightarrow V(f) \geqslant V(g)$. Obviously, if a representing functional exists, then the preference relation is a weak order.

A classical example of a representing functional is Savage's (1954) subjective expected utility (SEU). *Subjective expected utility* holds if a preference relation can be represented by the functional

$$SEU(f) = \int_S U(f(s))dP(s),$$

where $U : I\!R \to I\!R$ is the utility function and P is an (additive) probability measure on $\mathcal{A}$. Utility is cardinal (i.e., it is unique up to scale and location) and the probability measure is unique. For a simple act $f = \sum_{i=1}^{n} x_i 1_{A_i}$ the above integral reduces to

$$SEU(f) = \sum_{i=1}^{n} U(x_i)P(A_i).$$

Since its introduction by Savage, many preference conditions describing SEU have been offered (e.g. Anscombe & Aumann, 1963; Wakker, 1984, 1989; d 'Aspremont & Gevers, 1990; Gul, 1992). Savage's framework in which SEU has been derived is now accepted as the natural way of modelling decision under uncertainty, and we have adopted that setup here.

A further example of a representing functional is Choquet expected utility (CEU). This functional has been introduced by Schmeidler (1989, first version 1982) and generalized by Gilboa (1987). It extends SEU by allowing the probability measure to be non-additive. This so-called *capacity* v satisfies $v(S) = 1, v(\emptyset) = 0$, and $v(A) \geqslant v(B)$ if $A \supset B$ and $A, B \in \mathcal{A}$. A capacity v is *strictly monotonic* if $v(A) > v(B)$ for $A \supsetneqq B$ and $A, B \in \mathcal{A}$.

Choquet expected utility holds if the preference relation can be represented by the functional

$$CEU(f) = \int_{I\!R_+} v(\{s \in S | U(f(s)) \geqslant \tau\}) d\tau + \int_{I\!R_-} [v(\{s \in S | U(f(s)) \geqslant \tau\}) - 1] d\tau.$$

For a simple act $f = \sum_{i=1}^{n} x_i 1_{A_i}$, such that $U(x_i) \geqslant U(x_{i+1})$ for $i = 1, \ldots, n-1$, CEU can be written as

$$CEU(f) = \sum_{i=1}^{n} U(x_i)[v(\cup_{j=1}^{i} A_j) - v(\cup_{j=1}^{i-1} A_j)].$$

Derivations of CEU have further been provided by Wakker (1989), Nakamura (1990), and Chew & Karni (1994). CEU-forms with linear utility are presented in Chateauneuf (1991), de Waegenaere and Wakker (2001), and Diecidue and Wakker (2002).

In terms of the underlying preference conditions the difference between these two representing functionals, SEU and CEU, is captured in the strength of the *sure-thing-principle*: $f_A h \succcurlyeq g_A h \Leftrightarrow f_A h' \succcurlyeq g_A h'$ for all involved acts. For SEU the full force of this principle is required, whereas for CEU the principle should hold only for acts which are pairwise *comonotonic* (f, g are comonotonic if there exists no states s, s' such that $f(s) > f(s')$ and $g(s) < g(s')$). The weakened version of the sure thing principle has been called comonotonic independence in Chew & Wakker (1996). We will use the term *comonotonic sure thing principle* here.

The central model in this paper is cumulative prospect theory (CPT). Under CPT a key role is assigned to the status quo outcome, which for simplicity of exposition is assumed to be the zero outcome. Outcomes are then perceived as deviations from the status quo, hence as gains or losses. For a given act f we define the gain-part f^+ as the act "f with all losses $f(s) < 0$ replaced by the status quo" and the loss part f^- as the act "f with all gains $f(s) > 0$ replaced by the status quo." The act f can then be viewed as the statewise sum of f^+ and f^-. *Cumulative Prospect Theory* holds if the representing functional for $\succcurlyeq$ has the form

$$CPT(f) = CEU^+(f^+) + CEU^-(f^-),$$

where CEU^+ is a CEU-form depending on a capacity v^+, and CEU^- is a CEU-form depending on a capacity v^-. The capacities are uniquely determined under CPT, and the utility is a ratio scale as it is fixed at the status quo: $U(0) = 0$. If in the above equation we use instead of the capacity v^-, the dual capacity $\hat{v}^-(\cdot) := 1 - v^-(S\backslash\cdot)$, then we can write

$$CPT(f) = \int_{\mathbb{R}_+} v^+(\{s \in S | U(f(s)) \geqslant \tau\})d\tau + \int_{\mathbb{R}_-} [\hat{v}^-(\{s \in S | U(f(s)) \leqslant \tau\})]d\tau.$$

For a simple act $f = \sum_{i=1}^{n} x_i 1_{A_i}$, such that $U(x_i) \geqslant U(x_{i+1})$ for $i = 1, \ldots, n-1$, and for some $k \in \{0, \ldots, n\}$ indicating the number of gain outcomes of the act f (i.e.,

$U(x_k) \geqslant 0 > U(x_{k+1})$), CPT can be written as

$$\begin{aligned} CPT(f) &= \sum_{i=1}^{k} U(x_i)[v^+(\cup_{j=1}^{i} A_j) - v^+(\cup_{j=1}^{i-1} A_j)] \\ &\quad + \sum_{i=k+1}^{n} U(x_i)[v^-(\cup_{j=1}^{i} A_j) - v^-(\cup_{j=1}^{i-1} A_j)]. \end{aligned}$$

The latter functional has been introduced by Starmer and Sugden (1989). Axiomatizations with general utility have appeared in Luce and Fishburn (1991), Luce (1991), Tversky and Kahneman (1992), and Wakker and Tversky (1993). Derivations of CPT with specific forms for the utility function (linear/exponential, power, and variants of multiattribute utility) have been provided in Zank (2001) generalizing Wakker and Zank (2002). All these functional forms, including SEU and CEU, are special cases of the cumulative utility functional presented in Chew and Wakker (1996).

In the remainder of the paper we concentrate on a special case of CPT, where utility is linear. More precisely, the utility function will have the form

$$U(x) = \begin{cases} x, & \text{if } x \geqslant 0, \\ \lambda x, & \text{if } x \leqslant 0, \end{cases}$$

where the loss aversion parameter λ is positive.

Preference conditions are proposed to characterize CPT with linear utility, which we refer to as LCPT. The new condition, called independence of common increments, entails sign-dependence and the comonotonic sure thing principle, and moreover it implies linearity of utility on the gain domain and separately on the loss domain. The preference conditions are introduced in the next section for the finite states case. These results are then extended in Section 4 to simple acts on general state spaces, and finally in Section 5 some technical conditions are employed for the case of general acts. Possible extensions are presented in Section 6.

3 Finite State Spaces

Assume that the state space S is finite. That is $S = \{s_1, \ldots, s_n\}$ for a natural number $n \geqslant 3$. Acts $f = (f(s_1), \ldots. f(s_n))$ can be identified with the Cartesian product space $I\!R^n$. Hence, in this section we refer to acts as vectors $(f_1, \ldots, f_n)$ (f_i is short notation for $f(s_i)$). An act f is *rank-ordered* if its outcomes are ordered from best to worst: $f_1 \geqslant \cdots \geqslant f_n$. For each act f there exists a permutation ρ of $\{1, \ldots, n\}$ such that $f_{\rho(1)} \geqslant \cdots \geqslant f_{\rho(n)}$, i.e. such that the outcomes are *rank-ordered with respect to* ρ. For each permutation ρ of $\{1, \ldots, n\}$ the set $I\!R^n_\rho$ consists of those acts which are rank-ordered with respect to ρ. For example, if $\rho = id$ (i.e. $\rho(i) = i$ for all i), then $I\!R^n_{id}$ is the set of rank-ordered acts. Acts from a rank-ordered set $I\!R^n_\rho$ are obviously comonotonic.

The preference relation $\succcurlyeq$ on $I\!R^n$ satisfies *monotonicity* if $f \succ g$ whenever $f_i \geqslant g_i$ for all states s_i with a strict inequality for at least one state. By employing this condition we exclude null states, that is, states where the preference is independent of the magnitude of outcomes. Formally, a state s_i is *null* if $x_i f \sim y_i f$ for all acts f and all outcomes x, y.

The continuity condition defined here is continuity with respect to the Euclidean topology on $I\!R^n$: $\succcurlyeq$ satisfies *continuity* if for any act f the sets $\{g \in I\!R^n | g \succcurlyeq f\}$ and $\{g \in I\!R^n | g \preccurlyeq f\}$ are closed subsets of $I\!R^n$.

We now introduce the main condition in the paper. *Independence of common increments* holds if for any two acts $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ and $x \in I\!R$ we have

$$\begin{aligned}(f_1, \ldots, f_i, \ldots, f_n) &\succcurlyeq (g_1, \ldots, g_i, \ldots, g_n) \Rightarrow \\ (f_1, \ldots, f_i + x, \ldots, f_n) &\succcurlyeq (g_1, \ldots, g_i + x, \ldots, g_n),\end{aligned}$$

whenever $f_i, f_i + x, g_i, g_i + x$ are of the same sign (that is, either they are all gains or they are all losses), and all involved acts are pairwise comonotonic (that is, they are all from the same set of rank-ordered acts $\mathbb{R}^n_\rho$).

Independence of common increments says that a common absolute change of an outcome of the same rank does not reverse the preference between two acts as long as this change is not large enough to affect the rank or the sign of the outcomes. For x small enough, repeated application of this principle on acts containing only gains (or only losses) yields $(f_1 + x, \ldots, f_n + x) \succcurlyeq (g_1 + x, \ldots, g_n + x)$, indicating that it implies a weakened variant of the concept of constant absolute risk aversion (CARA), which could be called sign-dependent CARA. The restrictions on x mentioned above are crucial for the difference to traditional CARA. The principle, however, is stronger as sign-dependent CARA used in Zank (2001) for the derivation of CPT with linear/exponential utility, as the exponential form is excluded.

One can show that repeated application of independence of common increments implies local additivity on sets of pairwise comonotonic acts having the same number of gain outcomes. Therefore, there exist outcomes $x_1, \ldots, x_n$ such that

$$\begin{aligned}(f_1, \ldots, f_i, \ldots, f_n) &\succcurlyeq (g_1, \ldots, g_i, \ldots, g_n) \Rightarrow \\ (f_1 + x_1, \ldots, f_n + x_n) &\succcurlyeq (g_1 + x_1, \ldots, g_n + x_n)\end{aligned}$$

if $f_k, g_k \geqslant 0 > f_{k+1}, g_{k+1}$ and all acts are pairwise comonotonic. This shows that the property comes close to additivity on rank ordered sets. Such a condition has been used by Weymark (1981) to derive the generalized Gini welfare functions. The condition has been termed comonotonic additivity in de Waegenaere and Wakker (2001) and Diecidue and Wakker (2002). Our condition here is weaker because of its reference- and sign-dependent nature. If we would drop the sign- and the rank-dependence restrictions we would get additivity on general sets. That and monotonicity are equivalent to the nonexistence of a Dutch book, a condition used by de Finetti (1931) to derive subjective expected utility with linear utility. This demonstrates that the only features we have added to additivity are rank-dependence, reference-dependence, and sign-dependence, the basic characteristics of CPT.

In the following we show that independence of common increments is a necessary condition for CPT with linear utility, and that it implies the comonotonic sure thing principle. We present the results and corresponding proofs in the main text to further clarify the nature of this principle.

LEMMA 1 *If LCPT holds for $\succcurlyeq$ on $I\!R^n$ then independence of common increments is satisfied.*

PROOF: We prove the lemma for the case that acts are rank-ordered. The remaining cases are similar. Hence, suppose that $(f_1, \ldots, f_i, \ldots, f_n) \succcurlyeq (g_1, \ldots, g_i, \ldots, g_n)$ with $f, g \in I\!R^n_{id}$. Assume that there exists x such that $(f_1, \ldots, f_i + x, \ldots, f_n), (g_1, \ldots, g_i + x, \ldots, g_n) \in I\!R^n_{id}$ and that $f_i, f_i + x, g_i, g_i + x$ have the same sign, say they are gains . Then, substituting LCPT we get

$$\begin{aligned} & \sum_{j=1}^{k} f_j[v^+(\{s_1, \ldots, s_j\}) - v^+(\{s_1, \ldots, s_{j-1}\})] \\ & + \sum_{j=k+1}^{n} \lambda f_j[v^-(\{s_1, \ldots, s_j\}) - v^-(\{s_1, \ldots, s_{j-1}\})] \\ \geqslant \; & \sum_{j=1}^{k'} g_j[v^+(\{s_1, \ldots, s_j\}) - v^+(\{s_1, \ldots, s_{j-1}\})] \\ & + \sum_{j=k'+1}^{n} \lambda g_j[v^-(\{s_1, \ldots, s_j\}) - v^-(\{s_1, \ldots, s_{j-1}\})]. \end{aligned}$$

Adding on both sides of the inequality above $x[v^+(\{s_1, \ldots, s_i\}) - v^+(\{s_1, \ldots, s_{i-1}\})]$ gives the desired result. Note that in the above summands k and k' may differ, showing that act f may contain a different number of outcomes which are gains than act g. In the case that $f_i, f_i + x, g_i, g_i + x$ are all losses we add $\lambda x[v^-(\{s_1, \ldots, s_i\}) - v^-(\{s_1, \ldots, s_{i-1}\})]$ on both sides of the inequality. Hence, independence of common increments holds. $\square$

In earlier derivations of CPT complex independence condition have been used. In Tversky and Kahneman (1992), and Wakker and Tversky (1993) the conditions is termed sign-comonotonic tradeoff consistency. Luce and Fishburn (1991) and Luce (1991) use a condition called compound gamble and joint receipt. Our principle does not immediately imply these conditions, however, it does this in the presence of the remaining preference conditions as is shown in Theorem 3 below. In the following lemma we show that the comonotonic sure thing principle is implied by independence of common increments.

LEMMA 2 *Assume that $\succcurlyeq$ on $I\!R^n$ is a weak order satisfying continuity. Then independence of common increments implies the comonotonic sure thing principle.*

PROOF: We prove the lemma assuming that all acts are from $I\!R^n_{id}$. For acts from $I\!R^n_\rho$, where ρ is an arbitrary permutation of $\{1, \ldots, n\}$, the proof is complicated only by the tedious indexing of outcomes, otherwise results are derived in a similar fashion. Suppose $f, g \in I\!R^n_{id}$, such that $f_i = g_i = h_i$, and

$$h_i f \succcurlyeq h_i g (\Leftrightarrow f \succcurlyeq g).$$

Clearly, $\min\{f_{i-1}, g_{i-1}\} \geqslant h_i$ if $i \in \{2, \ldots, n\}$ and $h_i \geqslant \max\{f_{i+1}, g_{i+1}\}$ if $i \in \{1, \ldots, n-1\}$. We show that the outcome h_i can be replaced with any h'_i satisfying $\min\{f_{i-1}, g_{i-1}\} \geqslant h'_i$ if $i \in \{2, \ldots, n\}$ and $h'_i \geqslant \max\{f_{i+1}, g_{i+1}\}$ if $i \in \{1, \ldots, n-1\}$ without changing the above preference. Suppose, that $i \in \{2, \ldots, n\}$ and that $0 > \min\{f_{i-1}, g_{i-1}\}$ $(\geqslant h'_i)$. Then applying independence of common increments with $x = h'_i - h_i$ gives

$$h_i f \succcurlyeq h_i g \Leftrightarrow h'_i f \succcurlyeq h'_i g.$$

Similarly, if $i \in \{1, \ldots, n-1\}$ and we have $(h'_i \geqslant) \max\{f_{i+1}, g_{i+1}\} > 0$, then applying independence of common increments with $x = h'_i - h_i$ gives

$$h_i f \succcurlyeq h_i g \Leftrightarrow h'_i f \succcurlyeq h'_i g.$$

Suppose now that for $i \in \{2, \ldots, n-1\}$ we have $\min\{f_{i-1}, g_{i-1}\} \geqslant 0 \geqslant \max\{f_{i+1}, g_{i+1}\}$ (or for $i = 1$ we have $0 \geqslant \max\{f_{i+1}, g_{i+1}\}$, or for $i = n$ we have $\min\{f_{i-1}, g_{i-1}\} \geqslant 0$). Then, using continuity we find that independence of common increments can first be applied with $x = -h_i$ implying

$$h_i f \succcurlyeq h_i g \Leftrightarrow 0_i f \succcurlyeq 0_i g,$$

and a second application of the principle with $x' = h'_i$ gives

$$0_i f \succcurlyeq 0_i g \Leftrightarrow h'_i f \succcurlyeq h'_i g.$$

This shows, that "state-wise" the comonotonic sure thing principle holds (also called comonotonic coordinate independence in Wakker, 1989), and by appropriate successive application of it one can show that the comonotonic sure thing principle holds for general events A as opposed to single states $\{s_i\}$. □

We can now present the main result of this section.

THEOREM 3 *Suppose that* $\succcurlyeq$ *is a preference relation on* $I\!R^n$, *for* $n \geqslant 3$. *Then the following two statements are equivalent:*

(i) LCPT holds with strictly monotonic capacities v^+, v^-.

(ii) The preference relation $\succcurlyeq$ *is a monotonic continuous weak order satisfying independence of common increments.* □

The proof of this theorem is presented in the appendix. Further remarks on how the preference conditions can be weakened are postponed until Section 6.

4 Simple Acts

In the previous section we have introduced preference conditions characterizing LCPT in the case that the state space is finite. These results can easily be extended for more general state spaces, and this is the purpose of the present section. As a first step we reformulate our preference conditions to accommodate general state spaces. In this section S, the set of states of the world, can be finite or infinite. We focus on acts $f \in \mathcal{F}^s$ of the form $f = \sum_{i=1}^n x_i 1_{A_i}$, for a partition $\{A_1, \ldots, A_n\}$. When there is no confusion we avoid the explicit mentioning of the particular partition.

The preference relation $\succcurlyeq$ on $\mathcal{F}^s$ satisfies *monotonicity* if $f_A x \succ f_A y$ whenever $x > y$ and the event A is non-null. The definition of a null event is the natural extension of the definition of a null state: an event B is *null* if $x_B f \sim y_B f$ for all acts f and all outcomes x, y.

In this section continuity is also defined with respect to the Euclidean topology: $\succcurlyeq$ satisfies *simple-continuity* if for any simple act $f = \sum_{i=1}^n x_i 1_{A_i}$ the sets $\{(y_1, \ldots, y_n) \in I\!R^n | \sum_{i=1}^n y_i 1_{A_i} \succcurlyeq f\}$ and $\{(y_1, \ldots, y_n) \in I\!R^n | \sum_{i=1}^n y_i 1_{A_i} \preccurlyeq f\}$ are closed subsets of $I\!R^n$.

Independence of common increments holds if for any two simple acts f and g which can be represented using the same partition $\{A_1, \ldots, A_n\}$, any event A_i from this partition, and any outcome $x \in I\!R$ we have

$$\begin{aligned} f_{A_i} y &\succcurlyeq g_{A_i} z \Rightarrow \\ f_{A_i}(y + x) &\succcurlyeq g_{A_i}(z + x), \end{aligned}$$

whenever $y, z, y + x, z + x$ are of the same sign (that is either they are all gains or they are all losses), and all involved acts are pairwise comonotonic.

We can now formulate the main result of this section:

Theorem 4 *Suppose there exists three disjoint non-null events. Then, the following two statements are equivalent for a preference relation $\succcurlyeq$ on the set of simple acts $\mathcal{F}^s$:*

(i) LCPT holds,

(ii) the preference relation $\succcurlyeq$ is a simple-continuous monotonic weak order that satisfies independence of common increments. □

The proof of Theorem 4 is presented in the Appendix.

5 General Result

In the previous section we have introduced the axioms describing LCPT for a preference relation on the set of simple acts. In this section we extend the functional derived in Theorem 4 to the set of all acts, $\mathcal{F}$. To do this it is not necessary to extend all properties of $\succcurlyeq$ on $\mathcal{F}^s$ to hold on the entire set of acts $\mathcal{F}$. It turns out that independence of common increments and monotonicity need to hold only for $\succcurlyeq$ on $\mathcal{F}^s$ if we employ an appropriate continuity condition. The idea behind this is to exploit the specific structure of the set of acts $\mathcal{F}$.

The distance between two acts f, g, measured in the *supnorm* is defined as $\sup_{s\in S}|f(s)-g(s)|$. We say that $\succcurlyeq$ is *supnorm-continuous* if for each act f the sets $\{g \in \mathcal{F}|g \succcurlyeq f\}$ and $\{g \in \mathcal{F}|g \preccurlyeq f\}$ are closed sets under the supnorm. That supnorm-continuity is not very restrictive follows from the fact that it equivalent to Euclidean continuity on $I\!R^n$, and further by the fact that it is equivalent to continuity of the utility function under SEU, CEU, and general CPT.

The next lemma shows that weak ordering and supnorm-continuity ensures the existence of a *certainty equivalent* for each act, i.e., a constant act $x(f)$ such that $x(f) \sim f$. Then we exploit the well-known fact that $\mathcal{F}^s$ is a supnorm-dense subspace of $\mathcal{F}$: the existence of a certainty equivalent $x(f)$ to each act f allows us to define $LCPT(f)$ as the value of $LCPT(x(f))$ established in Theorem 4, and therefore the extension of LCPT from $\mathcal{F}^s$ to $\mathcal{F}$ is established.

LEMMA 5 *Suppose there exists three disjoint non-null events. Further, assume that* $\succcurlyeq$ *on* $\mathcal{F}$ *is a weak order that satisfies supnorm-continuity, and that LCPT holds on* $\mathcal{F}^s$*. Then each act* f *has a certainty equivalent* $x(f)$*.* □

The proof of the lemma is given in the appendix.

Take now any act f. Recall that f is bounded, such that there exist $x, y \in \mathbb{R}$ with $x \geqslant f(s) \geqslant y$ for all states $s \in S$. It is now easy to generate simple acts f^l, g^l (bounded by x, y from above and below, respectively) which converge in the supnorm, respectively, from above and below to f. This holds similarly for the corresponding certainty equivalents, so that the definition $LCPT(f) = LCPT(x(f))$ makes sense. Actually this argument would hold true on any subset $\mathcal{F}'$ of acts containing all simple acts, i.e., $\mathcal{F} \supseteq \mathcal{F}' \supseteq \mathcal{F}^s$. The theorem below summarizes the previous analysis in the main result of this section:

THEOREM 6 *Suppose there exists three disjoint non-null events. Then, the following two statements are equivalent for a preference relation* $\succcurlyeq$ *on the set of acts* $\mathcal{F}$*:*

(i) LCPT holds,

(ii) the preference relation $\succcurlyeq$ *is a supnorm-continuous weak order on* $\mathcal{F}$ *that satisfies monotonicity and independence of common increments on* $\mathcal{F}^s$*.* □

6 Concluding Remarks

The extensions described in this section are focusing on the results in Section 3. In Theorem 3 the continuity condition can be dropped if the existence of a certainty equivalent for each act is ensured. The technique to prove the result would be similar to the one used in Diecidue and Wakker (2002) by employing results of Aczel (1966). We have indicated that independence of common increments implies locally the comonotonic additivity used in Diecidue and Wakker. The proof here would be more complicated as one has to deal with the sign dependent nature of the independence principle, which initially implies local comonotonic additivity on rank-ordered subsets of $\mathbb{R}^n$ in which precisely k states have gain outcomes ($k = 0, \ldots, n$). On these subsets LCPT would hold and one needs to fit together the different LCPT-functionals in order to derive LCPT on all of $\mathbb{R}^n$. Once the result for finite spaces is established, the results for simple acts and those for general acts can similarly be derived without any continuity assumption. Again the existence of a certainty equivalent to each act is required.

Instead of dropping continuity one could, from a technical point of view, dispense of monotonicity. As stated in Section 3 the role of monotonicity was to ensure that all states are essential. A general consequence of monotonicity is that utility is increasing, hence representing LCPT functionals that agree on $\mathbb{R}^n_{id,+}$ with a functional

$$(f_1, f_2, \ldots, f_n) \mapsto \frac{1}{2}f_1 - \frac{1}{3}f_2 + \frac{1}{6}f_3,$$

where in a state (here s_2) an increase in an outcome leads to a decrease in utility, are avoided. Also, recall that we have introduced capacities as nonadditive but monotonic extensions of probability measures. Therefore, the marginal impact of an event for a capacity is nonnegative. By dispensing of monotonicity the marginal impact of an event may be negative, and this feature is unreasonable from an economic point of view.

Let us now focus on the principle of independence of common increments, which can be formulated more appealing from an empirical point of view. Many studies suggest that individuals pay comparably more attention to extreme outcomes (e.g. Lopes 1987, Gilboa 1988, Jaffray 1988, Cohen 1992), hence to worst and best outcomes. Suppose that S is a finite state space. For act f, outcome x, and event A denote $(f+x)_A f$ the act assigning $f_i + x$ for $s_i \in A$ and f_i for $s_i \notin A$. Independence of common increments implies that for any two acts f and g and $x \in I\!R$ we have

$$\begin{aligned} f &\succcurlyeq g \Rightarrow \\ (f+x)_A f &\succcurlyeq (g+x)_A g, \end{aligned}$$

whenever $f_i, f_i + x, g_i, g_i + x$ are of the same sign (that is either they are all gains or they are all losses) for $s_i \in A$, all involved acts are from the same set of rank-ordered acts $I\!R^n_\rho$, and, moreover, $A = \{s_{\rho(1)}, \ldots, s_{\rho(m)}\}$ or $A = \{s_{\rho(l)}, \ldots, s_{\rho(n)}\}$ for some $m, l \in \{1, \ldots, n\}$.

This version of the principle, which could be called *independence of common increments at extreme outcomes*, says that a preference between two acts remains unchanged if the best outcomes or the worst outcomes are increased or decreased by the same common outcome, if the original and the modified outcomes are all of the same sign, and all involved acts are pairwise comonotonic. To relate this condition to the earlier version in Section 3 note that the new condition actually is equivalent to independence of common increments, as the next lemma shows.

LEMMA 7 *Suppose $\succcurlyeq$ is a preference relation on $I\!R^n$. Then independence of common increments is equivalent to independence of common increments at extreme outcomes.* □

Given the result in this lemma, Theorem 3 holds if we replace independence of common increments by independence of common increments at extreme outcomes.

A final comment refers to our assumption that acts are bounded. We have restricted our analysis throughout the paper to such acts. Our result in Section 5 can be extended to unbounded acts by using a technique similar to the definition of integrals. Such techniques are discussed for example in Wakker (1993).

7 Appendix

PROOF OF THEOREM 3: First, statement (i) is assumed, and statement (ii) is concluded. Suppose LCPT holds for $\succcurlyeq$ on $\mathbb{R}^n$ with strictly monotonic capacities. Weak ordering is immediate from the existence of the representing functional for $\succcurlyeq$. Monotonicity holds because utility is increasing and the capacities are strictly monotonic. Continuity of utility implies continuity of $\succcurlyeq$. Independence of common increments holds by Lemma 1. This completes the proof of statement (ii).

Next, statement (ii) is assumed and statement (i) is derived. The proof consists of several intermediate results. First, it is shown that on the set of rank-ordered acts $\mathbb{R}^n_{id}$ the preference relation is represented by the additive function described in Lemma 8 below. Then (Lemma 9), it is shown that the additive function in Lemma 8 is a restriction of a LCPT functional. Lemma 10 indicates that similar results can be derived for $\succcurlyeq$ on $\mathbb{R}^n_\rho$, for any permutation ρ of $\{1,\ldots,n\}$. Then, it is shown that the different LCPT restrictions fit together into a general functional, such that LCPT holds for $\succcurlyeq$ on $\mathbb{R}^n$.

LEMMA 8 *The preference relation* $\succcurlyeq$ *on* $\mathbb{R}^n_{id}$ *is represented by the additive functional:*

$$(f_1,\ldots,f_n) \mapsto \sum_{i=1}^n V_i(f_i),$$

with continuous strictly increasing functions $V_1,\ldots,V_n : \mathbb{R} \to \mathbb{R}$*, which are uniquely determined satisfying* $V_i(0) = 0$ *for all* i *and* $\sum_{i=1}^n V_i(1) = 1$.

PROOF: The proof follows by combining different existing results. First note that by Lemma 2 the comonotonic sure thing principle holds. Then the result follows from

Corollary 3.5 in Wakker (1993). There, an additive functional representation is derived with cardinal functions V_i. By fixing $V_i(0) = 0$ for all i and $\sum_{i=1}^n V_i(1) = 1$, the statement in our lemma is derived. $\triangle$

Let now $k \in \{0, \ldots, n\}$ be arbitrarily fixed. We concentrate on acts $f \in I\!R^n_{id}$, having k gain outcomes, i.e., $f_k \geqslant 0 > f_{k+1}$. Independence of common increments is satisfied, so that for $x \in I\!R$

$$f \sim g \Rightarrow (f_i + x)_i f \sim (g_i + x)_i g,$$

whenever $f_i, f_i + x, g_i, g_i + x$ are of the same sign. Substitution of the additive functional derived in Lemma 8 gives

$$V_i(f_i) - V_i(g_i) = V_i(f_i + x) - V_i(g_i + x),$$

which implies, first locally and then by continuity globally, linearity of V_i on $I\!R_+$ if $i \leqslant k$ and on $I\!R_-$ if $i \geqslant k+1$. As k was chosen arbitrarily we conclude that the functions V_i are linear for gains and linear for losses. Continuity, monotonicity, and the fact that $V_i(0) = 0$ implies that the V_i's are of the form

$$V_i(x) = \begin{cases} \alpha_i^+ x, & \text{if } x \geqslant 0, \\ \beta_i x, & \text{if } x \leqslant 0, \end{cases}$$

with $\alpha_i^+ > 0, \beta_i > 0$ for all $i = 1, \ldots, n$. Moreover, $\sum_{i=1}^n V_i(1) = 1$ implies $\sum_{i=1}^n \alpha_i^+ = 1$, so that we can refer to the α_i^+'s as decision weights for gain outcomes. Let now $\sum_{i=1}^n V_i(-1) = -\lambda$ with λ positive. Then we can define $\alpha_i^- := \beta_i/\lambda$ as the decision weights corresponding to loss outcomes (they are nonnegative and their sum equals 1). Let us summarize:

Lemma 9 *There exist positive decision weights* α_i^+ *and positive decision weights* α_i^- *such that the preference relation* $\succcurlyeq$ *on* $\mathbb{R}^n_{id}$ *is represented by the additive functional:*

$$(f_1, \ldots, f_n) \mapsto \sum_{i=1}^{n} V_i(f_i),$$

with functions V_i *of the form*

$$V_i(x) = \begin{cases} \alpha_i^+ x, & \text{if } x \geqslant 0, \\ \lambda \alpha_i^- x, & \text{if } x \leqslant 0, \end{cases}$$

for a positive λ. △

In the preceding analysis we have restricted attention to acts $f \in \mathbb{R}^n_{id}$. It is easy to show that for acts $f \in \mathbb{R}^n_\rho$, where ρ is an arbitrary permutation of $\{1, \ldots, n\}$, similar results can be derived. The proof is complicated only by the more complex indexing of outcomes. We can conclude the following statement:

Lemma 10 *There exist positive decision weights* $\alpha_{i,\rho}^+$ *and positive decision weights* $\alpha_{i,\rho}^-$ *such that the preference relation* $\succcurlyeq$ *on* $\mathbb{R}^n_\rho$ *is represented by the additive functional:*

$$LCPT_\rho(f_1, \ldots, f_n) \mapsto \sum_{i=1}^{n} V_{i,\rho}(f_{\rho(i)}),$$

with functions $V_{i,\rho}$ *of the form*

$$V_i(x) = \begin{cases} \alpha_{i,\rho}^+ x, & \text{if } x \geqslant 0, \\ \lambda_\rho \alpha_{i,\rho}^- x, & \text{if } x \leqslant 0, \end{cases}$$

for a positive λ_ρ. △

It remains to show that the decision weights $\alpha_{i,\rho}^+$ and $\alpha_{i,\rho}^-$, and the loss aversion parameters λ_ρ are all independent of the permutation ρ.

First we show that λ_ρ is independent of ρ. If for a permutation ρ of $\{1, \ldots, n\}$ the set $\mathbb{R}^n_\rho \bigcap \mathbb{R}^n_{id}$ contains nonconstant acts, then $LCPT_\rho$ and $LCPT_{id}$ jointly represent the preference relation $\succcurlyeq$ on the intersection $\mathbb{R}^n_\rho \bigcap \mathbb{R}^n_{id}$. If for a permutation ρ of $\{1, \ldots, n\}$

the set $I\!R^n_\rho \bigcap I\!R^n_{id}$ contains only constant acts, then, as $n \geqslant 3$, it follows that there exists a permutation σ of $\{1, \ldots, n\}$ such that the set $I\!R^n_\rho \bigcap I\!R^n_\sigma$ contains nonconstant acts and also $I\!R^n_\sigma \bigcap I\!R^n_{id}$ contains nonconstant acts. As the LCPT-representations are uniquely determined, it follows that the loss aversion parameters λ_ρ are independent of ρ, hence equal to λ. Obviously, the different LCPT-representations agree on the subset of constant acts which are commonly contained in each set $I\!R^n_\rho$. Using continuity one can show that each act f has a *certainty equivalent*, i.e., a constant act $x(f) \sim f$. It then follows for acts $f \in I\!R^n_\rho, g \in I\!R^n_\sigma$ that

$$f \succcurlyeq g \Leftrightarrow x(f) \succcurlyeq x(g)$$

implying

$$LCPT_\rho(f) = LCPT(x(f)) \geqslant LCPT(x(g)) = LCPT_\sigma(g).$$

This shows that the different functionals $LCPT_\rho$, agree on common domain and are restrictions of one general LCPT functional representing $\succcurlyeq$ on $I\!R^n$.

Next we determine the capacities v^+ and v^-. Let A be any event. We consider the act 1_A0 assigning outcome 1 for state $s \in A$ and 0 for $s \notin A$. By setting $v^+(A) := LCPT(1_A0)$ we define a capacity on S for gains, satisfying

$$\alpha^+_{i,\rho} = v^+(\{s_{\rho(1)}, \ldots, s_{\rho(i)}\}) - v^+(\{s_{\rho(1)}, \ldots, s_{\rho(i-1)}\})$$

for $i = 1, \ldots, n$ and any permutation ρ. This definition makes sense because for different premutations ρ, σ, the respective LCPT representations agree on common domain, hence the corresponding decision weights are equal. Because the decision weights are unique the same is true for the capacity v^+. Moreover, as all decision weights are positive, it follows that the capacity is strictly monotonic.

By setting $v^-(A) := LCPT(-1_A 0)/\lambda$ we define a second capacity on S now for losses, satisfying

$$\alpha^-_{i,\rho} = v^-(\{s_{\rho(1)}, \ldots, s_{\rho(i)}\}) - v^-(\{s_{\rho(1)}, \ldots, s_{\rho(i-1)}\})$$

for $i = 1, \ldots, n$ and any permutation ρ. The capacity for losses v^- is also unique, and strictly monotonic.

We can now conclude that $\succcurlyeq$ on $I\!R^n$ is represented by an LCPT functional with unique capacities v^+, v^-, and unique loss aversion parameter λ. This concludes the proof of Theorem 3. □

PROOF OF THEOREM 4: That statement (i) implies statement (ii) is immediate from the definition of LCPT. We assume statement (ii) and prove statement (i). Suppose that a partition $\{A_1, \ldots, A_n\}$ is given. Let $\mathcal{F}^s_{\{A_1,\ldots,A_n\}}$ denote the set of simple acts of the form $\sum_{i=1}^n x_i 1_{A_i}$, for outcomes x_i. Obviously, the set $\mathcal{F}^s_{\{A_1,\ldots,A_n\}}$ can be identified with $I\!R^n$ (or $I\!R^m$ if precisely $m \geqslant 3$ events in the partition $\{A_1, \ldots, A_n\}$ are non-null), and further the restriction of $\succcurlyeq$ to $\mathcal{F}^s_{\{A_1,\ldots,A_n\}}$ is a weak order that satisfies monotonicity, continuity, and independence of common increments. Hence, statement (ii) of Theorem 3 holds and we conclude that LCPT holds on $\mathcal{F}^s_{\{A_1,\ldots,A_n\}}$ with capacities $v^+_{\{A_1,\ldots,A_n\}}, v^-_{\{A_1,\ldots,A_n\}}$. Note that these – in fact restrictions of – capacities may not be strictly monotonic if some events in the partition $\{A_1, \ldots, A_n\}$ are null.

The above arguments can be repeated for any fixed partition of S, and it remains to show that these different LCPT-functionals are restrictions of a general LCPT functional representing $\succcurlyeq$ on $\mathcal{F}^s$.

It is well known that given two simple acts $f = \sum_{i=1}^n x_i 1_{A_i}$, and $g = \sum_{j=1}^m y_j 1_{B_j}$, there exists a partition $\{C_{i,j}\}_{i=1,j=1}^{n,m}$, which is a common refinement of both $\{A_1, \ldots, A_n\}$, and

$\{B_1, \ldots, B_m\}$, such that f, g can be represented as simple acts with respect to the same partition. One can for example define $C_{i,j}$ as the intersection of the events A_i and B_j. Then, $f = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i 1_{C_{i,j}}$, and $g = \sum_{i=1}^{n} \sum_{j=1}^{m} y_j 1_{C_{i,j}}$. Suppose that $LCPT_{\{A_1,\ldots,A_n\}}$ represents $\succcurlyeq$ on $\mathcal{F}^s_{\{A_1,\ldots,A_n\}}$ and $LCPT_{\{B_1,\ldots,B_m\}}$ represents $\succcurlyeq$ on $\mathcal{F}^s_{\{B_1,\ldots,B_m\}}$. Let further $LCPT_{\{C_{i,j}\}_{i=1,j=1}^{n,m}}$ represent $\succcurlyeq$ on $\mathcal{F}^s_{\{C_{i,j}\}_{i=1,j=1}^{n,m}}$. As $\mathcal{F}^s_{\{A_1,\ldots,A_n\}}, \mathcal{F}^s_{\{B_1,\ldots,B_m\}}$ are both included in $\mathcal{F}^s_{\{C_{i,j}\}_{i=1,j=1}^{n,m}}$ it follows that $LCPT_{\{C_{i,j}\}_{i=1,j=1}^{n,m}}$ represents $\succcurlyeq$ on $\mathcal{F}^s_{\{A_1,\ldots,A_n\}}$ and on $\mathcal{F}^s_{\{B_1,\ldots,B_m\}}$. We conclude that the different LCPT-functionals are restrictions of a general LCPT-functional representing $\succcurlyeq$ on $\mathcal{F}^s$. Hence statement (i) of the theorem follows.

□

Proof of Lemma 5: Let $f \in \mathcal{F}$. Because all acts are bounded there exist $x, y \in I\!R$ such that $x \geqslant f(s) \geqslant y$ for all states s. We can construct (similar to the classical derivation of the Lebesgue integral) two sequences of simple acts f^l, g^l converging in the supnorm from above and below, respectively, to f. These sequences can be chosen such that

$$x \geqslant f^l(s) \geqslant f^{l+1}(s) \geqslant f(s) \geqslant g^{l+1}(s) \geqslant g^l(s) \geqslant y$$

holds for each state s. As LCPT holds on $\mathcal{F}^s$ it follows that

$$LCPT(x) \geqslant LCPT(f^l) \geqslant LCPT(g^l) \geqslant LCPT(y)$$

and hence

$$x \succcurlyeq f^l \succcurlyeq g^l \succcurlyeq y$$

for all l. By supnorm-continuity and the fact that the two sequences converge to f it follows that $x \succcurlyeq f \succcurlyeq y$. In particular it follows that the sets $\{z \in I\!R | z \succcurlyeq f\}$ and $\{z \in I\!R | z \preccurlyeq f\}$ are closed and have a nonempty intersection. They contain at least one element $x(f) \sim f$ with

$$LCPT(x(f)) = \begin{cases} x(f), & \text{if } x(f) \geqslant 0, \\ \lambda x(f), & \text{if } x(f) \leqslant 0. \end{cases}$$

Therefore, we can also conclude that in the intersection of these two sets there exists a unique element $x(f) \sim f$, the certainty equivalent of f. $\square$

Proof of Lemma 7: First we assume that independence of common increments holds. Let $f, g \in I\!R^n_\rho$, such that $f \succcurlyeq g$. Suppose that $A = \{s_{\rho(1)}, \ldots, s_{\rho(m)}\}$ for some $1 \leqslant m \leqslant \min\{k, k'\}$, where k, k' denote the number of gain outcomes of f, g, respectively. We have to show that for any $x \in I\!R$ such that $(f + x)_A f, (g + x)_A g \in I\!R^n_\rho$ it follows that $(f + x)_A f \succcurlyeq (g + x)_A g$. For $x = 0$ there is nothing to show. If $x > 0$, then independence of common increments can repeatedly be applied to $s_{\rho(1)}$, then to $s_{\rho(2)}$, etc., until $s_{\rho(m)}$, such that, by induction, we get:

$$f \succcurlyeq g \Rightarrow (f + x)_A f \succcurlyeq (g + x)_A g.$$

If $x < 0$, independence of common extremes can repeatedly be applied starting with $s_{\rho(m)}$ then to $s_{\rho(m-1)}$, etc., until $s_{\rho(1)}$, such that, by induction, we get $f \succcurlyeq g \Rightarrow (f+x)_A f \succcurlyeq (g+x)_A g$. Note here, that $f_{\rho(m)} + x, g_{\rho(m)} + x$ must be positive in order to apply independence of common increments.

The proof follows similarly for $A = \{s_{\rho(l)}, \ldots, s_{\rho(n)}\}$ for some $\max\{k, k'\} < l$. Therefore, independence of common increments at extreme outcomes holds.

For the reversed implication, assume that independence of common increments at extreme outcomes holds. Let $i \in \{1, \ldots, n\}$. We have to show that for any $x \in I\!R$ such that $f, g, (f + x)_{\rho(i)} f, (g + x)_{\rho(i)} g \in I\!R^n_\rho$ and $f_{\rho(i)}, f_{\rho(i)} + x, g_{\rho(i)}, g_{\rho(i)} + x$ are of the same sign it follows that

$$f \succcurlyeq g \Rightarrow (f+x)_{\rho(i)} f \succcurlyeq (g+x)_{\rho(i)} g.$$

If $f_{\rho(i)}, g_{\rho(i)}$ are gains then we apply independence of common increments at extreme outcomes first with x and $A = \{s_{\rho(1)}, \ldots, s_{\rho(i)}\}$ and then with $-x$ and $A' = \{s_{\rho(1)}, \ldots, s_{\rho(i-1)}\}$ and get

$$\begin{aligned} f &\succcurlyeq g \Rightarrow (f+x)_A f \succcurlyeq (g+x)_A g \\ &\Rightarrow (f-x)_{A'}(f+x)_A f \succcurlyeq (g-x)_{A'}(g+x)_A g \\ &\Leftrightarrow (f+x)_{\rho(i)} f \succcurlyeq (g+x)_{\rho(i)} g. \end{aligned}$$

Note that in order to maintain comonotonicity of $f, g, (f+x)_{\rho(i)} f, (g+x)_{\rho(i)} g$, the outcome x must be chosen such that $f_{\rho(i-1)} \geqslant f_{\rho(i)} + x$ and $g_{\rho(i-1)} \geqslant g_{\rho(i)} + x$. Under these conditions the above applications of independence of common increments at extreme outcomes are well defined.

If $f_{\rho(i)}, g_{\rho(i)}$ are losses then we apply independence of common increments at extreme outcomes first with x and $A = \{s_{\rho(i)}, \ldots, s_{\rho(n)}\}$ and then with $-x$ and $A' = \{s_{\rho(i+1)}, \ldots, s_{\rho(n)}\}$. It follows that independence of common increments holds, as i, and ρ were arbitrary. This completes the proof of the lemma. □

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